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STRENGTH AND DESIGN OF METAL BEAM-COLUMNS

By Walter J. Austin,[1] F. ASCE

SYNOPSIS

This paper contains a review of the state of knowledge relating to the behavior, strength, and design of metal members subjected to combined axial compression and bending as of 1961. Practical procedures are presented for the prediction of the strength of individual members under known loading and supporting conditions, and then these procedures extended, when possible, to apply to assemblages of members. Recent developments which should make possible a more accurate analysis and a more economical design are described. Although primarily a review of published work, this paper also contains some original interpretations and some re-casting of previously published work to give a unified treatment to the subject.

INTRODUCTION

Beam-columns are members which are subjected to combined axial compression and bending. The bending may arise from lateral loads, couples applied at any point on the beam, or from end moments resulting from eccentricity of the axial loads at one or both ends of the member. This classification covers a large range of members. At one extreme, when the effects of the bending forces are small relative to the effects of the axial load, the beam-column is essentially a concentrically loaded column, and at the other extreme,

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[1] Prof. of Civ. Engrg. Rice Univ., Houston, Texas.

when the effects of the axial compressive forces are small, the member is essentially a beam. Thus, in effect, both columns and beams are special cases of the general classification of beam-columns.

BEAM-COLUMNS WHICH BUCKLE IN ONE PLANE

An analysis of the strength and behavior of beam-columns which, when subjected to combined compressive axial loads and bending forces acting in one plane, fail by buckling in the plane of the applied bending forces, substantially without twisting will be presented in this section. The subject in this part, therefore, applies to columns which if subjected to only concentric compression would also fail by buckling in the plane considered, without twisting. Both open and closed cross section columns are considered.

Strength of Individual Members.—Ultimate Strength Theory.—The determination of the ultimate load of a member subjected to combined compressive axial load and bending forces is essentially a non-linear stability problem in which the effects of inelastic action must be considered. The strain distribution through the depth is assumed to be linear for both elastic and inelastic conditions. The beam-column collapses when, because of yielding caused by the combined bending and axial stresses, the stiffness of the member against additional bending is reduced to the point where instability occurs, that is, the ultimate axial load is the critical load of the partially yielded column. The beam-column collapses by bending in the plane of the bending forces without twisting. This description of the behavior of a beam-column was given first by Theodor von Karman, Hon. M. ASCE (1).[2] F. Bleich (21) and S. Timoshenko (5) have analyzed the major contributions by von Karman, Ros and Brunner, Chwalla, Westergaard and Osgood, Jezek, and others. Studies by O. M. Sidebottom and M. E. Clark, M. ASCE in 1958 (37), R. L. Ketter, A. M. ASCE, E. L. Kaminsky, A. M. ASCE, and Lynn S. Beedle, F. ASCE in 1955 (27) and T. V. Galambos, A. M. ASCE, and Ketter in 1959 (39) represent further developments and verifications of the general approach originated by von Karman.

There is no doubt that the von Karman theory is correct and that the action of a member under combined axial compression and bending may be computed up to collapse provided that the stress-strain diagram of the material, initial conditions of residual stress and crookedness, and the boundary conditions are known. However, the computations are complex and laborious. It is not practical to use this exact theory directly to predict the strength of members except possibly for research studies.

Practical Considerations.—A simple method with which to estimate or predict the strength of individual beam-columns supported in a given manner and subjected to known loadings is necessary as a basis for the prediction of the strength of such members in structural frameworks and assemblies. Two simple approximate procedures have been developed. The first procedure, chronologically, is based on the assumption that failure occurs at the computed load which causes the initiation of yielding in the most highly stressed fiber. The second, more recent, approach uses empirical formulas based on test results and exact analyses.

Limiting Stress Criterion.—A satisfactory estimate of the strength of a structural beam-column can be found on the basis, or assumption, that failure

[2] Numbers in parentheses refer to corresponding items in Appendix I—Bibliography.

occurs when the computed value of the combined axial and bending stress in the most highly stressed fiber reaches the yield point or yield strength of the material. That is, the initiation of yielding is the criterion of failure. The bending action is assumed to be elastic up to the point of yielding. Therefore, a conventional elastic stress analysis may be made to determine the load at failure for any member, prismatic or non-prismatic, for any support conditions, and for any axial and lateral loading. The wide range of applications which can be considered without difficulty make this procedure quite attractive and useful.

In the application of the limiting stress criterion to structural-steel columns, it is necessary to take into consideration the initial imperfections of the member in order to obtain reasonable values of strength for beam-columns subjected to axial loads and relatively small bending forces, and to obtain a smooth transition from this case to the case of concentric axial loading. The imperfections which appear to have the greatest influence are initial crookedness and residual stress. The effects of all imperfections can, approximately, be taken into account in either of the following ways:

1. By assuming the existence of an equivalent initial eccentricity of application of the axial load, or equivalent initial crookedness; or
2. the flexural stiffness of the column can be assumed to be reduced in proportion to the reduction of the tangent modulus which corresponds to the average stress P/A.

Procedure 1 is well-known; procedure 2 represents an extension of the tangent modulus procedure for columns, frameworks, etc., to beam-columns, and has not been proposed previously, to the writer's knowledge. Both procedures are satisfactory from a practical standpoint although only semi-rational. Both procedures give a reasonable, smooth transition over the complete range of possible loadings from pure compression to pure bending, provided that the failure criterion for pure bending is based on a limiting yield stress, as is customary for elastic design procedures.

If the equivalent initial eccentricity procedure is adopted, it is assumed that the column is initially straight and free from defects, but that the axial load is applied at each end with an initial eccentricity, e_o, which is chosen because when no other bending forces act, the computed average stress, P/A, at initial yielding is equal to the strength of the concentrically loaded column. The values of the equivalent initial eccentricity ratio, $e_o\ c/r^2$, which correspond to the column strength values recommended by the Column Research Council (CRC) (45) for ASTM A-7 structural steel are given in Table 1.

If the initial crookedness procedure is adopted, it is assumed that the axial load is applied perfectly concentrically at the ends, but that the column is initially crooked, with initial deflections in the shape of a half sine wave, with a maximum deflection at midheight of Δ_o. The values of the initial crookedness ratio, $\Delta_o\ c/r^2$, for which the computed average stresses at initial yielding are equal to the column strength values recommended by the CRC for ASTM A-7 structural steel are also given in Table 1.

In the tangent modulus procedure, the column is assumed to be perfectly straight and the axial load is assumed to be applied concentrically, but the flexural stiffness of the column at any stress level is taken as $E_t\ I$, in which E_t is the tangent modulus, a function of the average stress, P/A. The tangent modulus function is assumed to be the same as that which applies to a concen-

trically loaded column of the same material. Therefore, this procedure is entirely consistent with the tangent modulus procedure for the prediction of the strength of concentrically loaded columns.

The values of tangent modulus which correspond to the column strength values recommended by the CRC (45) for ASTM A-7 steel are given by

$$E_t = E \qquad \text{for } \sigma_a \leq 0.50\,\sigma_y \quad \ldots\ldots\ldots\ldots \quad (1a)$$

and

$$\frac{E_t}{E} = 4\left[1 - \frac{\sigma_a}{\sigma_y}\right]\frac{\sigma_a}{\sigma_y} \qquad \text{for } \sigma_a \geq 0.50\,\sigma_y \quad \ldots\ldots\ldots\ldots \quad (1b)$$

in which σ_a is the average compressive stress, P/A, σ_y denotes the yield point, equal to 33 ksi, and E represents Young's modulus, equal to 29,000 ksi. Note that whereas the equivalent initial eccentricity, or crookedness, ratio is a function of the slenderness ratio, L/r, only the reduction in stiffness in the tangent modulus procedure is a function of the average stress level, P/A.

TABLE 1.—INITIAL ECCENTRICITY AND CROOKEDNESS RATIOS[a,b]

	Slenderness Ratio, L/r								
	0	20	40	60	80	100	120	140	160
Column Strength, σ_c, ksi	33.0	32.6	31.5	29.6	26.9	23.5	19.3	14.6	11.2
$e_o\ c/r^2$	0	0.011	0.038	0.067	0.078	0.060	0.016	ϵ[c]	ϵ[c]
$\Delta_o\ c/r^2$	0	0.011	0.040	0.073	0.090	0.072	0.021	ϵ[c]	ϵ[c]

[a] Corresponding to Column Strength values recommended by Column Research Council for ASTM A-7 structural steel.
[b] Yield Point = 33.0 ksi E = 29000 ksi
[c] ϵ is a small quantity

The strength values computed by the tangent modulus procedure agree well with those computed by the equivalent initial eccentricity procedure. In Fig. 1 are shown curves of average compressive stress at initial yielding versus L/r for eccentrically loaded, simply supported columns of ASTM A-7 structural steel computed by both procedures. The values of $e_o\ c/r^2$ given in Table 1 were used for the computations for the equivalent initial eccentricity procedure, and the values of tangent modulus were obtained from Eqs. 1. Curves are given for applied eccentricity ratios, $e\ c/r^2$, of 0.1, 0.5, 1.0, and 3.0. These values represent a wide range of combined loadings. It can be seen that the tangent modulus procedure gives consistently higher strengths than the equivalent initial eccentricity procedure. The maximum difference is in the order of 6%, which can be regarded as small.

Some allowance for initial imperfections and for non-linearity of the stress-strain diagram must be made for aluminum members and for members of other materials that have a proportional limit well below the yield strength. For such cases the tangent modulus procedure appears to be most applicable.

The tangent modulus procedure appears to be more useful than the equivalent initial eccentricity, or crookedness procedure because it is easier to use

for a wide variety of loadings, support conditions, and column configurations. It is more applicable for complex structures, such as frameworks, and to a wide variety of materials. It is more consistent in concept with the tangent modulus procedure for concentrically loaded columns which is being so widely adopted (as of 1961).

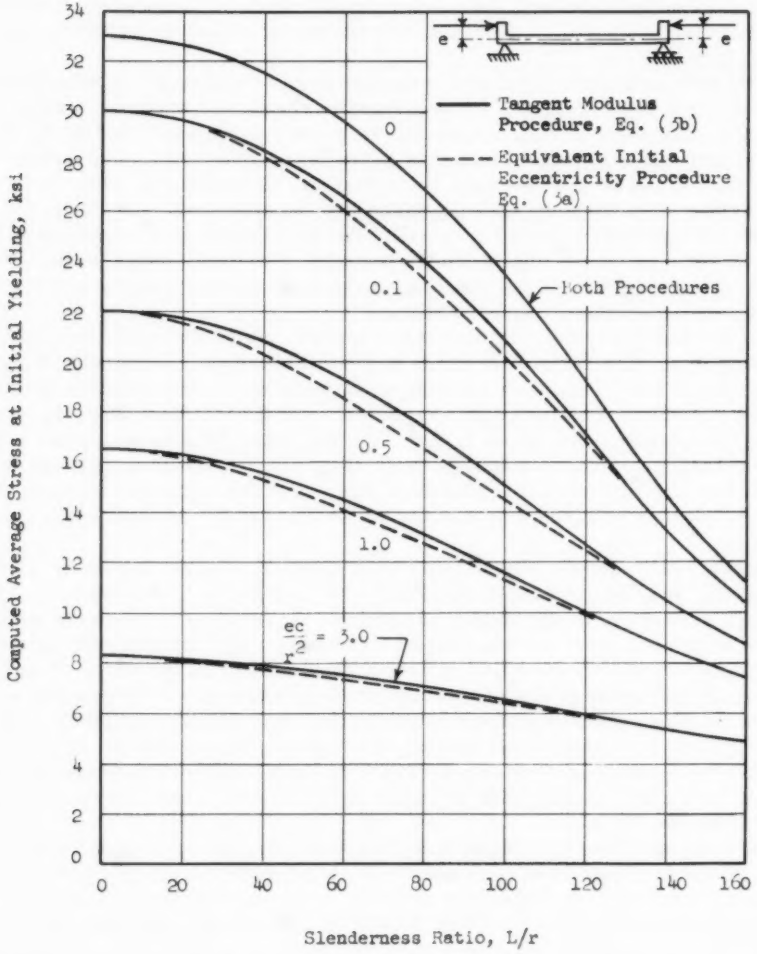


FIG. 1.—COMPARISON OF AVERAGE STRESSES AT INITIAL YIELDING COMPUTED BY TANGENT MODULUS AND INITIAL ECCENTRICITY PROCEDURES FOR SIMPLY SUPPORTED, ECCENTRICALLY LOADED STEEL COLUMNS

The limiting stress criterion of failure is compatible and consistent with elastic working stress design concepts which have formed the basis for American specifications, such as the American Railway Engineers Association

(AREA), and American Association of State Highway Officials (AASHO) for many years.

Experimental Verification of Limiting Stress Criterion.—The validity and accuracy of the initial yield criterion as a practical measure of the strength of structural beam-columns which buckle in one plane without twisting may be judged by the correlation between the theoretical ultimate loads and the maximum test loads which have been determined in tests of large scale members. Such correlations will be presented for the major American experimental investigations.

The results of three extensive investigations of the strength of eccentrically loaded, steel beam-columns are presented in Fig. 2. All specimens were tested with knife edge or roller supports which gave hinged ends in the direction of bending and practically fixed ends in the normal direction. The eccentricities were equal at the two ends. The loads at first yielding were computed from the secant formula. No allowance for imperfections was made, but for these tests this effect would be small. The concentration of experimental effort on tests of eccentrically loaded columns with equal eccentricities at the ends is a result of the fact that this is the most severe loading condition to which a beam-column may be subjected.

The results of tests on typical built-up steel column sections are presented in Fig. 2(a). The tests were made at the University of Wisconsin, Madison, Wis., under the direction of the Special Committee on Steel Column Research of the American Society of Civil Engineers (ASCE) (3), (4). With two exceptions every test result shown in Fig. 2(a) was obtained with an applied eccentricity ratio, $e\,c/r^2$, of 0.50. In these tests the maximum test loads agree rather well with the computed loads at initial yielding. The agreement is close for the slender columns, whereas for the short, stocky columns, more scatter exists.

In Fig. 2(b), the results of tests by Richard E. Mason, Gordon P. Fisher, F. ASCE, and George Winter, F. ASCE, (38) and Bruce G. Johnston, F. ASCE, and L. T. Cheney, F. ASCE (11) are shown. In these tests the cross sections were such that under combined axial compression and bending, the maximum compressive stress was uniform over a considerable flange width. These results are indicative of the behavior of beam-columns of wide flange and I-type sections which are bent in the strong direction and which are supported or restrained so that they neither deflect laterally nor twist. A wide range of eccentricity ratios was considered. The correlation between the maximum test loads and the computed loads at initial yielding is excellent for the results of both investigations.

The results of tests of steel wide flange-section columns loaded eccentrically so as to induce bending in the weak direction are presented in Fig. 2(c). The majority of these tests were made by Johnston and Cheney (11) at Lehigh University, Bethlehem, Pa. Wide ranges of slenderness and eccentricity ratios were investigated. The correlation between maximum test loads and computed loads at initial yielding was not good for short and medium length members, but was fair for slender members with L/r ratios of 90 or over.

In both the ASCE Special Committee investigation and in that of Mason, Fisher, and Winter, it was observed in many tests that flaking of mill scale or other observations indicated localized yielding at loads considerably less than the computed load at initial yielding. This phenomenon probably is due primarily to the presence of residual stresses in the specimens. However, it

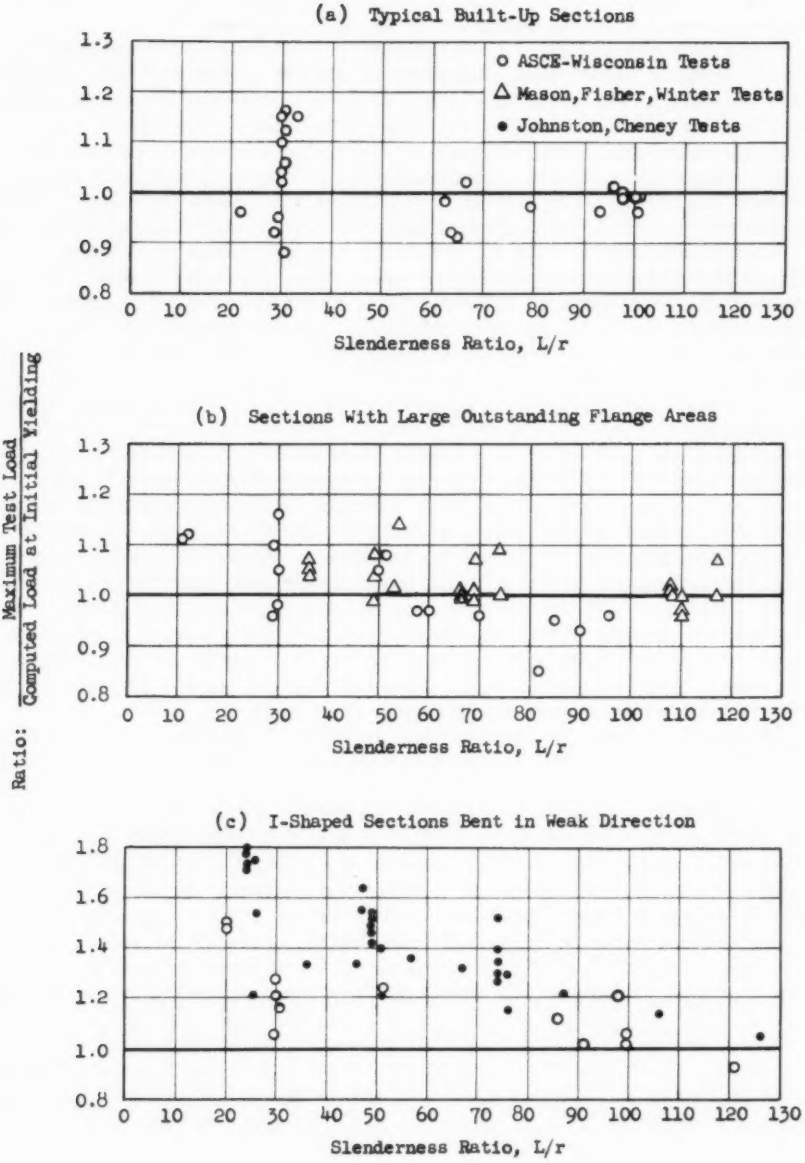

(a) Typical Built-Up Sections
o ASCE-Wisconsin Tests
△ Mason, Fisher, Winter Tests
• Johnston, Cheney Tests
Ratio: Maximum Test Load / Computed Load at Initial Yielding
Slenderness Ratio, L/r
(b) Sections With Large Outstanding Flange Areas
Slenderness Ratio, L/r
(c) I-Shaped Sections Bent in Weak Direction
Slenderness Ratio, L/r

FIG. 2.—COMPARISON OF MAXIMUM TEST LOADS WITH COMPUTED LOADS AT INITIAL YIELDING FOR ECCENTRICALLY-LOADED COLUMNS

should be emphasized that yielding ordinarily begins at loads less than the "computed load at initial yielding" and that this computed load by the limiting stress criterion is an approximation to the ultimate load.

It may be concluded from these large scale tests that the ultimate strength of well-designed, eccentrically loaded steel columns agrees well with the computed strength based on the yield point of the material and assuming elastic conditions up to the point of failure. The yield criterion for failure may considerably underestimate the ultimate strength of short beam-columns with unusually compact cross-sections, such as solid rectangular sections, wide flange sections bent in the weak direction, and so on. The yield criterion is most conservative for short members with large eccentricities or large lateral loads.

The limiting stress criterion of failure may be applied to beam-columns composed of materials other than low carbon structural steel. Tests of thirty-five small rectangular solid bars and tubes of structural aluminum have been reported by John W. Clark, M. ASCE (28). The tests were made with equal end eccentricities which produced buckling about the weak axis. These results indicate that the limiting stress criterion gives a reasonably satisfactory conservative prediction of the ultimate strength when the tangent modulus correction is made.

An important practical case is that of simply supported columns loaded with unequal end moments. Such loading is commonly called oblique loading. A small amount of test data is available, however, the Special Committee of ASCE gave, in their 1931 report (3), the results of tests on four columns subjected to thrust with equal and opposite eccentricities at the two ends. The columns failed by local plastic buckling of the flanges at the ends after yielding at loads which were 9% to 19% greater than the computed loads at initial yielding.

A number of tests of wide flange sections with oblique loading which produced bending in the strong direction have been made at Lehigh University (27), (20). For the tests in which the specimens did not fail by twisting, the yield criterion gave reasonable estimates of strength.

An elaborate analytical and experimental study of the strength of elastically restrained, eccentrically loaded columns has been conducted under CRC auspices by Paul B. Bijlaard, F. ASCE, Fisher, and Winter at Cornell University, Ithica, N.Y. (29). Tests are reported on eighteen columns, eight tests on square bar columns and ten tests on 4 I 9.5 lb columns bent in the weak direction.

In this investigation, it was found that the maximum test load often was greatly in excess of the computed load at first yielding. However, the ratios of maximum test load to computed load at initial yielding are only slightly higher for these tests with restrained ends than they were for the tests of hinged-ended, eccentrically loaded wide flange columns bent about the weak axis, which were previously presented. It should be noted that the end restraints behaved elastically at all loads, and therefore, as the column was weakened by yielding and axial compression, the end restraints assumed an increasingly greater role in preventing collapse. If the end restraints were to yield or be weakened as the axial load or moment increased, as would be likely in a practical structure, there is no doubt that the ultimate load would be reduced. One may conclude that the limiting stress criterion does not apply well to this test arrangement, but also this test arrangement does not correspond to most practical conditions.

Bijlaard, Fisher and Winter make the following observation:

> "In considering the merits of establishing ultimate loads as the basis for design criteria, it should be noted that when ultimate loads are reached, column deflections are excessively large, particularly for the larger

eccentricities. In contrast, these deflections are of moderate magnitude at those loads which cause incipient yielding."

Strength Analyses Based On Limiting Stress Criterion.—The maximum compressive stress in a member subjected to combined axial compression and bending in one plane may be computed by

$$\sigma_m = \frac{P}{A} + \frac{M'}{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

in which σ_m is the maximum combined compressive stress, P denotes the axial load, M′ represents the total bending moment, A is the cross-sectional area, and S is the section modulus. All of these quantities pertain to the most-highly-stressed cross section. The total moments, M′, due to the lateral loads, end moments, and the interaction of the axial loads and deflections, can be obtained for any given loading and supporting conditions by the solution of the appropriate differential equation or by numerical procedures such as the procedure proposed by Nathan Newmark, F. ASCE (12). One may always make such elastic analyses to determine the magnitude of a given loading which will cause the maximum stress in the member, σ_m, to reach the yield stress, σ_y, for any given distribution of loading and any supporting conditions. Many solutions have been derived and are available, expressed in convenient form. The following solutions are useful for practical conditions.

The average stress at the inception of virtual yielding in a simply supported, prismatic column subjected to axial loads applied at the ends with equal eccentricities, e, is given by the secant equation

$$\sigma_a = \frac{P}{A} = \frac{\sigma_y}{1 + (e + e_o)\frac{c}{r^2} \sec \frac{L}{2r}\sqrt{\frac{\sigma_a}{E}}} \quad \ldots\ldots\ldots \quad (3a)$$

in which r denotes the radius of gyration, c represents the distance from the extreme fiber in compression to the centroidal axis perpendicular to the plane of bending, and E is Young's modulus. In Eq. 3a, the equivalent initial eccentricity e_o is included to compensate for the imperfections in the column. If it is desired to use the tangent modulus correction for initial imperfections, the secant formula takes the form

$$\sigma_a = \frac{P}{A} = \frac{\sigma_y}{1 + \frac{e\,c}{r^2} \sec \frac{L}{2r}\sqrt{\frac{\sigma_a}{E_t}}} \quad \ldots\ldots\ldots\ldots \quad (3b)$$

in which the tangent modulus E_t is a function of σ_a, as previously stated. The secant formula forms the basis for the provisions in several design specifications.

An exact solution of a simply-supported column subjected to oblique eccentric forces has been given by D. H. Young, M. ASCE (6) and is presented fully by Timoshenko (5). The equations are not convenient to use directly but tables or curves based on these equations can be prepared. In fact, the AASHO design curves are based on these equations.

O. G. Julian, F. ASCE (26) presented to the CRC a convenient nomogram with which the maximum combined stresses and the location of the critical section can be found directly for a hinged-ended, prismatic compressed mem-

ber with equal or unequal end moments. In fact, the combined stresses at any desired section can be evaluated easily with this nomogram.

H. K. Stephenson, F. ASCE, and K. Cloninger, Jr., M. ASCE (22) give comprehensive formulas, tables, and charts for the evaluation of the virtual yield load in hinged-ended steel columns subjected to axial compressive loads applied with equal or unequal eccentricities at the ends. This work is extended to include restrained ends in a later paper by Stephenson (30). In both papers a full analysis of design procedures is given.

The formulas and procedures previously described each have a rather limited application and in many cases are awkward to evaluate. The following interaction formula gives only an approximate definition of the yield conditions, but it is of simple form and is applicable to prismatic columns with a wide range of practical supporting conditions and loadings:

$$\frac{\sigma_a}{\sigma_c} + \frac{\sigma_b}{\sigma_y \left(1 - \frac{0.65\,\sigma_a}{\sigma_e}\right)} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

(Interaction formulas of this type are quite common. This particular formula is, however, original.) In Eq. 4, σ_a is the average compressive stress, P/A, σ_b denotes the nominal maximum bending compressive stress, computed from the primary bending moments (that is, not including moments due to axial load), σ_c represents the average axial stress at collapse for the member when subjected to concentric compression, and σ_e is the elastic critical stress of the member when subjected to concentric compression. Thus

$$\sigma_c = \frac{\pi^2 E_t}{\left(\frac{K L}{r}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

and

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{K L}{r}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

in which E is Young's modulus and E_t denotes the tangent modulus which corresponds to the buckling stress σ_c. The quantity K L is the distance between points of inflection after buckling of the member of length L when subjected to concentric compression. The quantity K L is often called the "effective length" and K is called the "effective length factor." Practical means for the determination of the effective length factors for compression members have been presented (45), (44).

Consider the application of Eq. 4 to the case of a simply supported beam-column subjected to axial compressive load P applied with eccentricities, e, equal at both ends. In this case,

$$\sigma_b = \frac{P e c}{I} = \sigma_a \left(\frac{e c}{r^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and K = 1. The accuracy of Eq. 4 when applied to this problem has been investigated. The average stress σ_a has been computed from Eq. 4 with the corresponding value computed from the secant formula, Eq. 3b, for typical ASTM

A-7 structural steel with the properties given in Table 1 and in Eqs. 1, for values of the eccentricity ratio, e c/r^2, of 0.1, 0.5, 1.0, and 3.0, and for slenderness ratios, L/r, from 0 to 160. These comparisons are shown in Fig. 3. The average stresses at virtual yielding computed from Eq. 4 agree closely

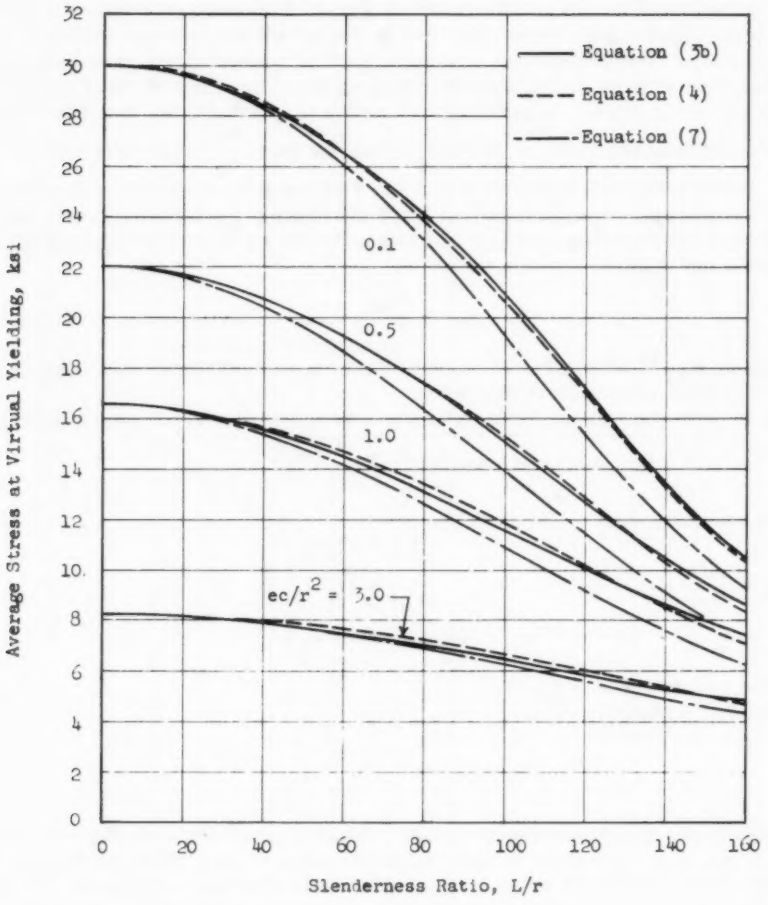


FIG. 3.—COMPARISON OF AVERAGE STRESSES AT VIRTUAL YIELDING COMPUTED BY EXACT SECANT EQUATION AND INTERACTION EQUATIONS (4) AND (7) FOR SIMPLY SUPPORTED, ECCENTRICALLY LOADED COLUMNS

with those computed by the secant formula for the wide range of slenderness and eccentricity ratios considered.

The following closely related interaction formula has also been investigated for the same case, the same material, and the same ranges of slenderness and

eccentricity ratios:

$$\frac{\sigma_a}{\sigma_c} + \frac{\sigma_b}{\sigma_y\left(1 - \frac{\sigma_a}{\sigma_e}\right)} = 1 \quad \ldots\ldots\ldots\ldots\ldots (7)$$

The solutions of Eq. 7 are also shown on Fig. 3. The average stresses at yielding agree closely with those computed by the secant formula for values of L/r of 40 or less. For more slender columns Eq. 7 is conservative, giving stresses as much as 17% low for $L/r = 160$. The major inaccuracy in Eq. 7 occurs in the first term, σ_a/σ_c. A rational correction is awkward, but this inaccuracy is counterbalanced in Eq. 4 by multiplying the term $\frac{\sigma_a}{\sigma_e}$ in the denominator of the second term by a factor of 0.65. Eq. 4 is much more accurate than Eq. 7.

Consider next the application of Eq. 4 to simply supported, prismatic members with other loadings. Eq. 4 can be applied to a wide variety of loadings by defining the nominal bending stress as

$$\sigma_b = \frac{M_{eq}}{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

in which M_{eq} is an equivalent value of uniform moment which has roughly the same effects as the actual moments.

The equivalent moments for uniform load and concentrated load at midspan (45a) are

for uniform load:
$$M_{eq} = \frac{M_m\left(1 + \frac{0.03\,\sigma_a}{\sigma_e}\right)}{\left(1 + \frac{0.23\,\sigma_a}{\sigma_e}\right)} \quad \ldots\ldots\ldots\ldots (9)$$

for concentrated load at midspan:
$$M_{eq} = \frac{M_m\left(1 - \frac{0.18\,\sigma_a}{\sigma_e}\right)}{\left(1 + \frac{0.23\,\sigma_a}{\sigma_e}\right)} \quad \ldots\ldots\ldots\ldots (10)$$

in which M_m is the maximum moment (at midspan) due to the lateral loads only. Eq. 4 can be used to determine the approximate yield conditions for hinged-ended members with any loading for which the bending moment is roughly symmetrical about mid-height with a maximum at or near mid-height, by taking M_{eq} equal to the maximum moment.

The analysis of simply supported members subjected to axial load and unequal end moments can be accomplished with Eqs. 4, 8, and 11. Consider that the end moments are M_1 and M_2, as shown in Fig. 4, and that M_1 is the greater in absolute value. The occurrence of virtual yielding in the interior part of the member may be determined by Eqs. 4 and 8. The occurrence of yielding at the ends may be determined by Eq. 11. Yielding will occur at the end when

$$\sigma_a + \frac{M_1}{S} = \sigma_y \quad \ldots\ldots\ldots\ldots\ldots (11)$$

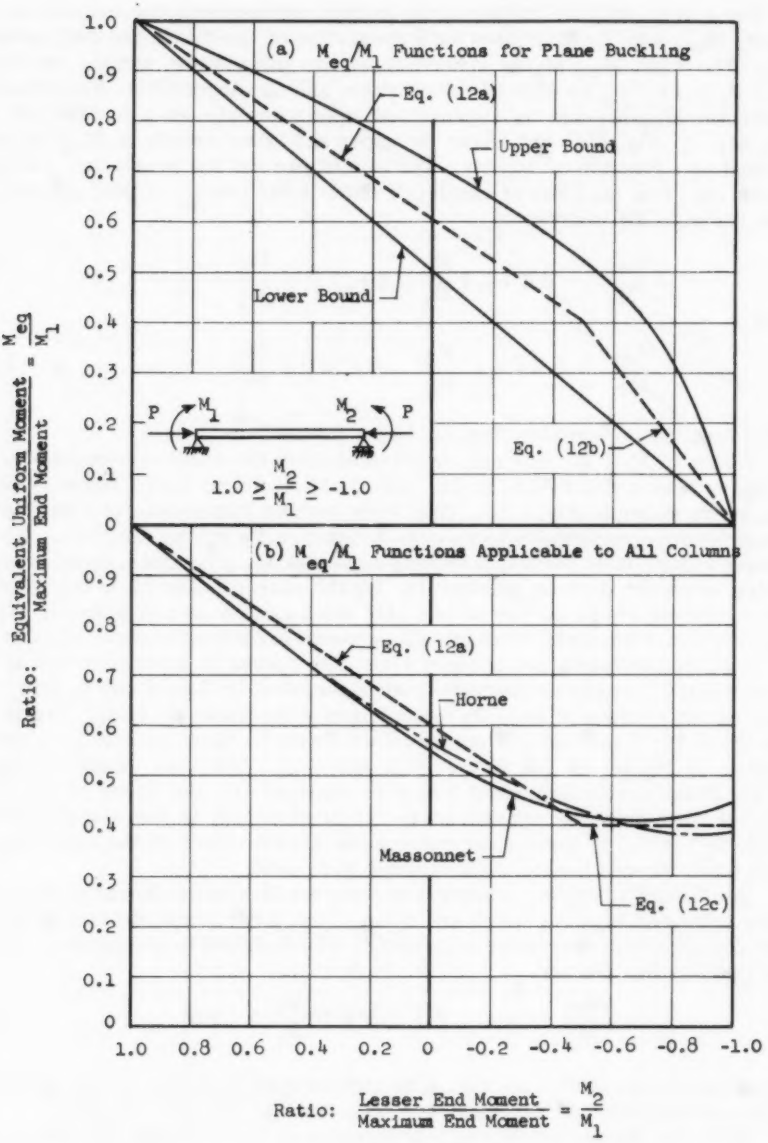


FIG. 4.—EQUIVALENT UNIFORM MOMENT FUNCTIONS FOR COLUMNS WITH AXIAL COMPRESSION AND UNEQUAL END MOMENTS

The lowest loading for which either Eq. 4 or 11 is satisfied is the loading which causes the initiation of virtual yielding.

For a beam-column subjected to unequal end moments, the equivalent moment, M_{eq}, may be expressed as a proportion of the maximum end moment, M_1. The ratio M_{eq}/M_1 is a function of two parameters, namely the ratios M_2/M_1 and σ_a/σ_e, as shown by Stephenson and Cloninger (22). Nevertheless, the ratio M_{eq}/M_1 can be represented, approximately, as a function of only M_2/M_1. In Fig. 4(a) are shown the upper and lower bounds of M_{eq}/M_1 when plotted as a function of M_2/M_1. The bounds are not far separated. Furthermore, the great majority of cases fall about in the center. A good approximation for M_{eq}/M_1 is given by

$$\frac{M_{eq}}{M_1} = 0.6 + 0.4\frac{M_2}{M_1} \quad \text{for } 1.0 \geq \frac{M_2}{M_1} \geq -0.5 \quad \cdots\cdots\cdots \quad (12a)$$

and

$$\frac{M_{eq}}{M_1} = 0.8 + 0.8\frac{M_2}{M_1} \quad \text{for } -0.5 \geq \frac{M_2}{M_1} \geq -1.0 \quad \cdots\cdots\cdots \quad (12b)$$

This function is plotted in Fig. 4(a) as a dashed line.

Values of M_{eq}/M_1 may also be obtained from the relationships proposed by Charles Massonnet, F. ASCE (35) (40) or M. R. Horne (34). These relationships are shown in Fig. 4(b). They were derived for the case of I-shaped columns loaded eccentrically to produce bending in the strong direction. The assumed failure is by combined bending and twisting. Therefore, these relationships were not derived specifically for the case considered in this section. Nevertheless, these curves of M_{eq}/M_1 when plotted as a function of M_2/M_1 fall within the bounds of the exact values except in the neighborhood of $(M_2/M_1) = -1$ where the values are conservative. The curves of average stress at virtual yielding versus slenderness ratio, computed by Eqs. 4 and 8 with these approximate values of M_{eq}/M_1 for columns with oblique eccentric loading, are shown in Fig. 5 for a single maximum eccentricity ratio $(e_1\, c/r^2) = 1$ and for values of M_2/M_1 of 1.0, 0.5, 0, -0.5, and -1.0. The exact curves computed from Julian's nomogram and Young's equations are also given in Fig. 5. It can be seen that the agreement is excellent except in the neighborhood of $(M_2/M_1) = -1$. In these computations the lowest values of the ratio M_{eq}/M_1 given by either of the two relationships were used.

For values of M_2/M_1 greater than zero, the Massonnet-Horne relationships give values of M_{eq}/M_1 which are close to the lower bound of exact values. A simpler, more conservative relationship which applies to all types of columns is given by Eq. 12a and

$$\frac{M_{eq}}{M_1} = 0.4 \quad \text{for } -0.5 \geq \frac{M_2}{M_1} \geq -1.0 \quad \cdots\cdots\cdots \quad (12c)$$

This function is plotted on Fig. 4(b) and the curves of average stress at virtual yielding obtained with Eqs. 4, 8, 12a, and 12c are shown in Fig. 5.

Since the Massonnet-Horne relationships or Eqs. 12a and 12c can be used for all classes of columns, and are thus general, they seem preferable to special equations for plane buckling, which though more accurate for the case of plane buckling, cannot be extended to cases where twisting and bending occur simultaneously.

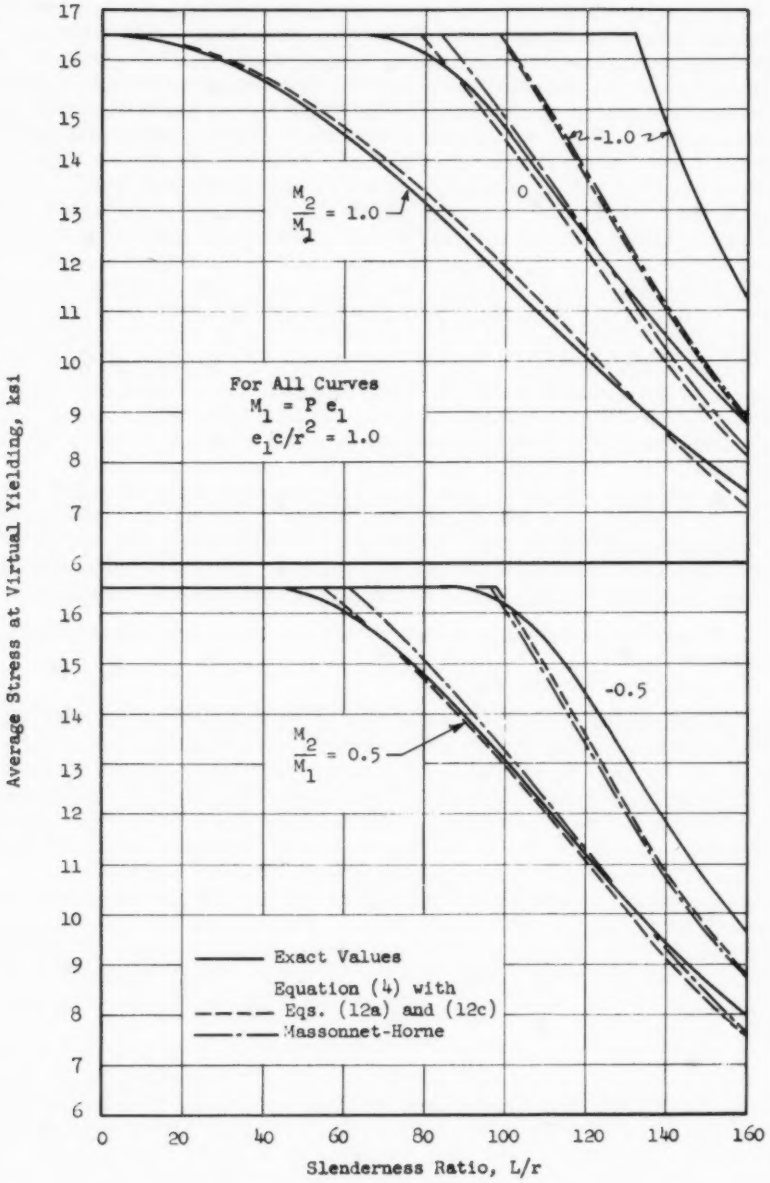


FIG. 5.—COMPARISON OF EXACT AVERAGE STRESSES AT VIRTUAL YIELDING WITH THOSE COMPUTED BY EQ. (4) WITH EQUIVALENT MOMENT PROCEDURE FOR SIMPLY SUPPORTED STEEL COLUMNS WITH OBLIQUE ECCENTRIC LOADING

Consider next prismatic beam-columns with other than simple supports. Consider cases where each end can be free, hinged, fixed, or elastically restrained both as to rotation and transverse translation. The member must have sufficient supports to guarantee overall stability, of course. Accurate formulas which are also simple and general have not been developed. However, it is conservative to use Eqs. 4 and 8 with $M_{eq} = M_m$, in which M_m is the maximum moment in the member determined by an ordinary structural analysis, without regard to the effects of axial load. Note that the correct value of the effective length, K L, appropriate for the support conditions, must be used in Eqs. 5 and 6. When the member is elastically restrained at the ends and when the axial buckling stress, σ_c, exceeds the proportional limit of the material, the effective length in Eq. 5 depends on the value of the tangent modulus. However, it is conservative to neglect this effect and to use in both Eqs. 5 and 6 the value of K L which is correct for axial buckling stresses below the elastic limit.

Strength Analyses Based On Inelastic Behavior.—Because of the trend toward ultimate strength design, formulas, graphs, and procedures for the prediction of the ultimate strength of beam-columns have been developed, based on either experimental data, theoretical inelastic solutions, or both. The theoretical solutions are usually based on the exact von Karman theory.

The results of extensive numerical computations of the ultimate carrying capacity of simply-supported, wide flange, steel columns have been presented by Galambos and Ketter (39). In this work it is assumed that failure occurs due to excessive bending in the plane of the web. The members are subjected to axial load and end moments. Two loading conditions are investigated:

1. Equal end moments applied in such a direction that they produce single curvature; and
2. moment at one end only.

The computations are exact for an 8 WF 31 section with a certain assumed typical residual stress pattern and a yield point of 33 ksi. It is considered that these results may be applied to all common wide flange column sections bent in the strong direction. Failure by combined bending and twisting or by local buckling has not been considered. The solutions are presented in the form of interaction diagrams. From the exact values approximate equations were developed to eliminate the need for interpolation. The theoretical values of buckling strength are compared with the results of the major pertinent experimental investigations, namely the tests of Mason, Fisher, and Winter (38), Johnston and Cheney (11), Massonnet (35), the Wisconsin ASCE series (3) and Lehigh tests. Good agreement is obtained for the tests in which twisting did not develop. The results of this study form the basis for the recommendations in an AISC manual (41).

Bijlaard, Fisher, and Winter (29) have presented an accurate analytical method and a more convenient simplified method for determining the buckling loads for columns with equal elastic rotational restraints at both ends and subjected to an axial compressive load applied with equal end eccentricities. The analytical procedures have been extended by Bijlaard (25) to cover beam-columns with arbitrary elastic end restraints and arbitrary end eccentricities. D. T. Wright (33) has proposed an interaction type strength equation for steel beam-columns, based on a rigorous theoretical plastic analysis of rectangular and I-shaped sections, for uniformly distributed and concentrated transverse

loads, and for simple and restrained ends. Both Bijlaard, Fisher, and Winter's procedure and Wright's formula agree well with test results.

The formulas and procedures previously stated are accurate but each has the disadvantage of a limited range of applicability. It appears that the following approximate interaction formula is a more practical formula for predicting the strength of metal beam-columns because of its simple form and general applicability:

$$\frac{\sigma_a}{\sigma_c} + \frac{\sigma_b}{\Gamma \sigma_y \left(1 - \frac{\sigma_a}{\sigma_e}\right)} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

In Eq. 13 σ_a is the axial stress at failure, σ_b denotes the nominal bending compressive stress at failure, σ_c and σ_e are the buckling stress and elastic critical stress for the column concentrally compressed and without external bending forces, as per Eqs. 5 and 6, and Γ is a plastic shape factor,

$$\Gamma = \frac{M_p}{M_y} = \frac{M_p}{S \sigma_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

in which M_p is the fully plastic moment capacity of the section for pure flexure without axial load. The shape factor Γ is always greater than unity. Typical values are as follows:

Wide flange section bending in plane of web , $\Gamma = 1.14$,

rectangular section , $\Gamma = 1.50$

wide flange section bending in plane of flanges, $\Gamma = 1.60$

There is an interesting possibility that Eq. 13 could be improved by replacing the term $\left(1 = \sigma_a/\sigma_e\right)$ in the denominator of the second term by $\left(1 - 0.65\ \sigma_a/\sigma_e\right)$, as in Fig. 4.

Consider the application of Eq. 13 to the prediction of the strength of simply supported beam-columns subjected to axial compressive load P applied with equal eccentricities at both ends. As previously, for this case,

$$\sigma_b = \frac{P\,e\,c}{I} = \sigma_a \left(\frac{e\,c}{r^2}\right) \quad \ldots\ldots\ldots\ldots\ldots (6)$$

and K = 1.

The ratios of maximum test load to maximum load computed by Eq. 13 are shown in Fig. 6 for two extensive investigations. In Fig. 6(a) are shown the results of tests by Mason, Fisher and Winter on hat-shaped sections with large outstanding flange areas. The results of these tests give an indication of the accuracy of Eq. 13 for the prediction of the strength of wide flange and I-type sections bent in the plane of the web and prevented from twisting and deflecting laterally. The agreement of the predicted and actual maximum loads is good. It should be noted, however, that the limiting stress criterion gave equally good correlation with these tests, as shown on Fig. 2(b). In Fig. 6(b) are shown the ratios of maximum test load to maximum load computed by Eq. 13 for Johnston and Cheney's tests of I-shaped sections bent in the weak direction. In computing the maximum loads by Eq. 13 for the specimens in Johnston and Cheney's tests, it was necessary to estimate σ_c as only a few tests were made

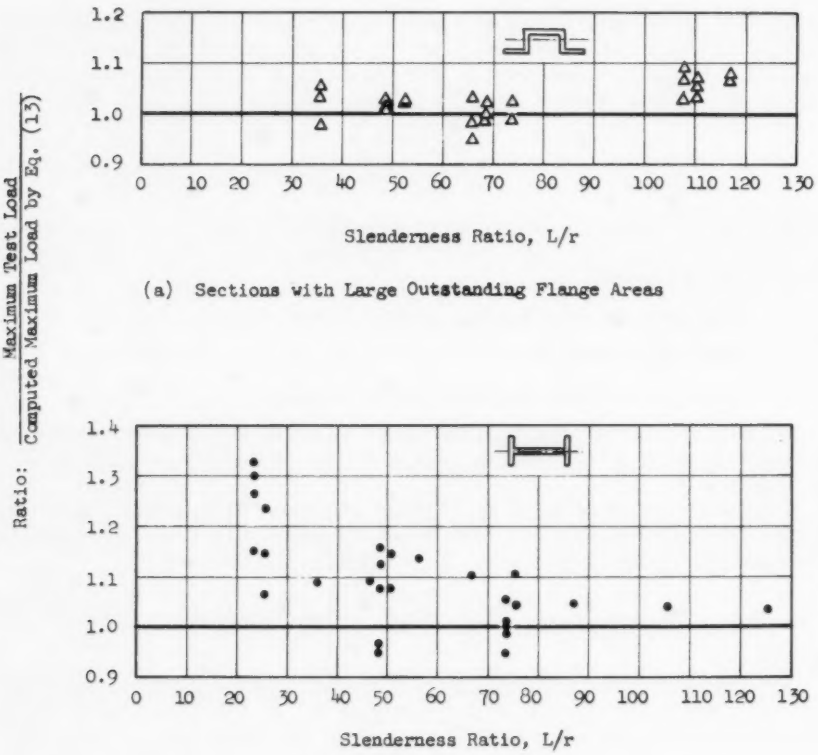
by the investigators with concentric loading. For this purpose the following formulas were used, as they seem reasonable and they give a reasonable fit with available data:

$$\sigma_c = \sigma_y \left(\frac{1 - 0.21\ \sigma_y}{\sigma_e}\right) \quad \text{for } 0.7 \leq \left(\frac{\sigma_c}{\sigma_y}\right) \leq 1.0 \quad \ldots\ldots \quad (15a)$$

and

$$\sigma_c = \sigma_e \qquad \text{for } 0 \leq \left(\frac{\sigma_c}{\sigma_y}\right) \leq 0.7 \quad \ldots\ldots\ldots \quad (15b)$$

The computed maximum loads are conservative for low slenderness ratios, but the correlation with test results is much better in this stocky column range



(a) Sections with Large Outstanding Flange Areas

(b) I-Shaped Sections Bent In Weak Direction

FIG. 6.—COMPARISON OF MAXIMUM TEST LOADS WITH COMPUTED MAXIMUM LOADS BY INTERACTION EQUATION (13) FOR SIMPLY SUPPORTED, ECCENTRICALLY LOADED STEEL COLUMNS

for the predictions of Eq. 13 than for those based on the limiting stress criterion, shown in Fig. 2(c).

Thus, Eq. 13 gives good correlation with the results of tests of eccentrically-loaded simply supported columns both for sections with large outstanding

flanges, such as for I-shaped beam-columns bending in the strong direction, and for compact sections, such as for I-shaped beam-columns bending in the weak direction. In each series of tests wide ranges of slenderness ratios and eccentricity ratios were considered. It may be concluded that Eq. 13 is quite accurate for the prediction of the strength of eccentrically-loaded, hinged, steel columns with any practical cross sections, slenderness ratios, and eccentricity ratios.

Clark (28) tested bars and tubes of structural aluminum. The tests were made with equal end eccentricities which produced buckling about the weak axis. He found excellent correlation with Eq. 13. In a later paper, Harry N. Hill, F. ASCE, Ernest C. Hartman, F. ASCE, and Clark (32) present the data from Clark's tests along with additional test data on H-section columns and round tubes. All of the data show an excellent agreement with Eq. 13. The test collapse loads are generally equal to or larger than the loads predicted by Eq. 13.

Herbert A. Sawyer, Jr., F. ASCE, in a study of Hill, Hartman, and Clark's paper (32), reviews the test data on duraluminum and chrome-molybdenum steel tubing obtained by Tuckerman, Petronko and Johnson using combined axial compression and third-point transverse loading. Sawyer shows that Eq. 13 safely and accurately predicts the strength of these columns. In this work Sawyer takes M_{eq} as the moment in the center third of the beam. Thus Sawyer shows that Eq. 13 may be applied to columns which have bending moments resulting from transverse loads.

The interaction equation Eq. 13 may also be used to predict the strength of simply supported beam-columns loaded with unequal end moments, or subjected to oblique eccentric loading. Consider that the end moments are M_1 and M_2, as shown in Fig. 4, and that M_1 is the greater in absolute value. For this case the member may buckle as a whole or a plastic hinge may develop first at the ends. Massonnet (40) has stated that Eq. 13 gives a good prediction of the ultimate forces when the member buckles as a whole, using his value for M_{eq}/M_1, as shown in Fig. 4(b). Therefore, any of the equivalent moment diagrams shown on Fig. 4(b) should be satisfactory.

A plastic hinge will form at the end acted on by the greater applied moment M_1 when this moment is equal to the fully plastic moment capacity of the member. The fully plastic moment capacity is not equal to M_p, but is reduced because of the axial compressive stress. The fully plastic moment capacity for members subjected to combined moment and axial compression M_{pc}, may be determined from the following approximate equations recommended by the Joint Welding Research Council (WRC)-ASCE Committee on Plasticity Related to Design (43). For strong axis bending of wide flange shapes,

$$M_{pc} = M_p \qquad \text{for } 0 \le \sigma_a \le 0.15\,\sigma_y \quad \text{(16a)}$$

and

$$M_{pc} = 1.18\left[1 - \left(\frac{\sigma_a}{\sigma_y}\right)\right] M_p \quad \text{for } 0.15\,\sigma_y \le \sigma_a \le \sigma_y \quad \text{(16b)}$$

for weak axis bending of wide flange shapes,

$$M_{pc} = M_p \qquad \text{for } 0 \le \sigma_a \le 0.4\,\sigma_y \quad \text{(16c)}$$

and

$$M_{pc} = 1.19\left[1 - \left(\frac{\sigma_a}{\sigma_y}\right)^2\right]M_p \quad \text{for } 0.4\,\sigma_y \leq \sigma_a \leq \sigma_y \ldots\ldots \quad (16d)$$

and for rectangular sections,

$$M_{pc} = \left[1 - \left(\frac{\sigma_a}{\sigma_y}\right)^2\right]M_p \ldots\ldots\ldots\ldots\ldots \quad (16e)$$

For convenience, Eqs. 15(a) and 15(b) are often used for all cross sections, as this simplification is conservative.

Massonnet (40) has compared the predictions given by the interaction equation (Eq. 13) with the exact theoretical values obtained by Galambos and Ketter (39) using the values of M_{eq} for oblique loading given by his formula. He found that for columns with equal end moments, the interaction equation is always conservative and quite accurate. For columns subjected to moment at one end only, the interaction equation is conservative for small values of the ratio $\sigma_b/\Gamma\,\sigma_y$ and slightly unconservative for large values of this ratio. Massonnet concludes that the interaction equation is adequate for this case. The interaction equation also gives predictions which agree well with the results of Massonnet's tests of wide flange columns bent in the strong direction by oblique eccentric loading.

Mason, Fisher, and Winter (38) have investigated the applicability of Eq. 13 to elastically restrained, eccentrically loaded columns. As shown in the paper by Bijlaard, Fisher, and Winter (29), such members of length L and eccentricity e may be analyzed by replacing the actual member with an equivalent simply supported, eccentrically loaded column of length K L and with an effective eccentricity e_e. The correct determination of K L and e_e involves the computation of a plastic reduction factor by a process of successive approximations. The predictions of the interaction equation, so modified, agree well (38) with the test results obtained by Bijlaard, Fisher, and Winter. Winter (24) has stated that it is safe, though less accurate, to use for K L the elastic effective length and to determine the portion of the total end moment resisted by the column, $P\,e_e$, by making an ordinary elastic analysis, neglecting the effects of the axial load in the analysis.

Strength of Frameworks.—The capacity of a structural system, or framework, may be limited by the strength of the component beam-columns. A simple method with which one can estimate, or predict, this capacity for any loading may be useful to evaluate the safety of existing structures under known loads to serve as the basis for the development of design procedures, or for direct use in design. In this analysis, the failure load of the structural system will be defined as the load which causes the first failure of a beam-column component.

The determination of the strength of a given indeterminate framework is essentially a trial and error procedure in which various loadings are assumed, and for each loading an analysis is performed to determine if the loading considered would cause failure. The failure criterion may be based on limiting stress concepts, in which case elastic analyses are appropriate, or on plastic strength concepts.

Limiting Stress Analyses.—Since up to failure the structure can be considered to act elastically, an elastic frame analysis can be performed to find the

bending moments at the ends of each member and the axial force in each member. Then the appropriate equations or methods previously examined for determining whether the capacity has been exceeded can be applied to each individual beam-column in the framework. In this way it can be determined if the strength of the framework has been exceeded.

For an accurate determination, it is necessary that the elastic frame analysis include the effects of axial loads. If the framework analysis is performed by moment distribution, or equivalent methods, it is necessary to use the proper values of fixed end moment, stiffness, and carry-over factor, which, for each member, are a function of the load parameter $P\,L^2/E_t\,I$. Also, to be consistent, in evaluating the stiffness of each member, one should use the tangent modulus E_t which corresponds to the average axial stress in the member. Tabulated values of the frame constants are readily available (46). It should be noted that in most civil engineering structures the axial compressions are not large enough to appreciably affect the frame constants. In such cases an ordinary frame analysis is suitable. In many cases simplifications are possible which involve only a small degree of approximation.

If the moments and axial compressive forces are found by a precise analysis, as outlined previously, the ability of each member to resist these forces can be found by application of the equations meant for simply supported members, provided that the axial force is less than the buckling force for the member considered as simply supported. Thus, for example, the interaction equation, (Eq. 4), with the equivalent moment modification for oblique loading, Eqs. 8 and 11, could be used.

If the axial loads appreciably affect the frame constants, the procedure outlined prior is laborious. A simpler, less accurate procedure has been suggested by Winter (24) and A. Y. Lee (14) for beam-columns in frameworks in which the joints cannot translate in the buckling action. First, an ordinary structural analysis is made, neglecting the effects of the axial compressions, to determine the primary end moments carried by the beam-columns. Then each beam-column is replaced by an equivalent simply-supported column whose length is equal to the effective length for concentric compression of the real, restrained column in the framework. Practical means for estimating the effective length are presented by Thomas C. Kavanagh, F. ASCE (44) and the CRC (45). This equivalent column may then be analyzed by any of the previously mentioned methods for compression plus the bending moments determined by the ordinary structural analysis. Thus, Eqs. 4, 8, and 11 could be used along with any of the equivalent moment relationships shown in Fig. 4(b). This procedure is conservative.

In the most general case where both translation and rotation of the joints can occur, yet an exact or semi-exact analysis is not feasible, it is suggested that an ordinary structural analysis be made, neglecting the effects of axial compression, to determine the primary end moments. Then Eq. 4 can be used to analyze each beam-column. In this computation the nominal bending stress σ_b is computed from the maximum bending moment. That is,

$$\sigma_b = \frac{M_m}{S} \qquad (17)$$

in which M_m is the maximum value of bending moment in the member determined from the ordinary structural analysis of the framework. The effective length in Eqs. 5 and 6 is determined for each member in the usual manner.

Inelastic Analyses.—To the writer's knowledge, no practical and general procedure has been developed for making an exact limit analysis of a complex framework where inelastic buckling is of major importance. However, Winter (24) suggests that an approximate procedure similar to that previously stated for the analysis of frameworks based on the limiting stress criterion can be applied. That is, an ordinary elastic frame analysis is made to determine the end moments and axial compressions and then Eq. 13 is used instead of Eq. 4 to analyze the individual members. If the joints cannot translate the equivalent moment procedure can be applied, with the end conditions determined by Eqs. 15 rather than Eq. 11. A study of work on the subject of inelastic frame stability was given by the Joint WRC-ASCE Committee (43).

BI-AXIAL BUCKLING WITHOUT TWISTING

This section considers the strength of torsionally-stiff beam-columns which are subjected to applied bending forces that cause either bending about the major principal axis only or about both principal axes. The beam-columns are free to deflect in all directions.

Primary Bending in Strong Direction.—The case of columns subjected to axial compression and lateral forces or moments acting to cause bending in the strong direction is one of the most frequently occurring cases of beam-columns. The typical column has essentially three ranges of behavior which are illustrated in Fig. 7.

In Fig. 7 the curve GDCN is the strength curve for the column loaded concentrically and buckling, as usual, in the weak direction, and the curve GHK is for the column loaded concentrically and buckling in the strong direction with lateral deflections prevented. The curve ABDE defines the strength of the column under eccentric loading about the major principal axis when buckling normal to the plane of the end moments is prevented. However, in the case under consideration buckling in the weak direction is not restrained. Therefore, for L/r values greater than at D, the average stress causing buckling in the weak direction, curve DCN, is less than the average stress causing buckling failure in the strong direction, as given by curve DE.

Hence, an upper limit to the true strength curve is given by curve ABDCN. This curve is an upper limit because it is based on the most favorable possible assumption that there is no coupling between the two effects. However, in all probability there is some coupling and the true strength curve probably looks like the solid line curve ABCN, with a smooth transition between curves AD and DN. The existence of the transition and its size, have not been well defined by experimental or theoretical work.

To summarize, short columns with large bending forces, for which the behavior is defined by curve AB, buckle essentially in the plane of the applied forces and develop as much strength as if they were restrained from deflecting in the weak direction. Long slender columns with small bending forces buckle in a direction normal to the plane of the bending forces and develop essentially the strength of the member loaded concentrically, that is, the bending forces cause no loss of strength. Intermediate length columns may buckle about both principal axes simultaneously and the strength is then less than the strength for buckling exclusively in one direction or the other.

Bi-Axial Bending.—Columns which are subjected to axial compression and bending moment acting about both principal axes bend simultaneously in both

directions and fail ultimately by plastic instability. Tests of six columns loaded with an eccentricity diagonal to the principal axes were reported by the ASCE Special Committee on Steel Column Research (4). The average stress at collapse was compared with the average stress at first yielding computed by the secant formula. The measured average stress at collapse was from 15% to 40% greater than the theoretical average stress at initial yielding, with an average excess of 26%. The writers of the report comment, "This is much larger than the values for one-direction eccentricity, and indicates that the effect of

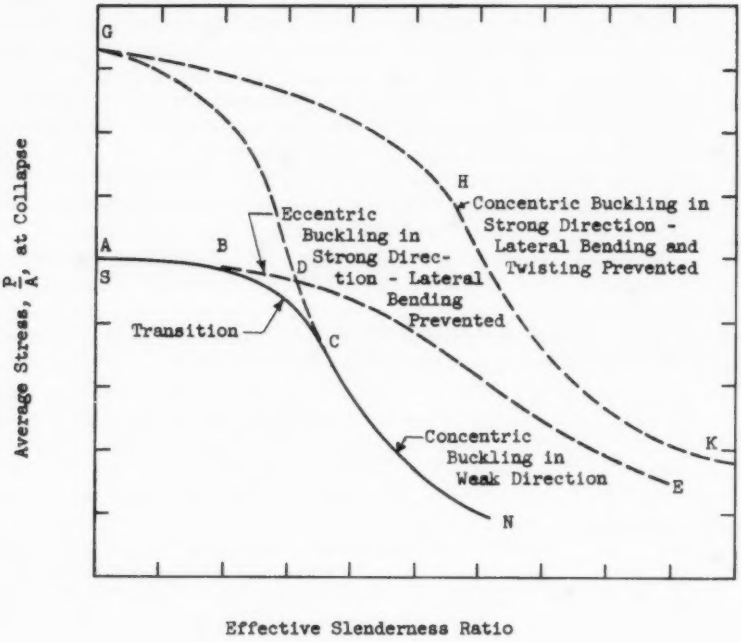


FIG. 7.—STRENGTH CURVE FOR TORSIONALLY STIFF COLUMN WITH ECCENTRICITY IN STRONG DIRECTION ONLY

eccentricity in both directions, which condition generally obtains in structures, may be discounted considerably in practice." These results are explainable on the grounds that the maximum fiber stress is highly localized as it occurs at only one corner of the section. This corner may obviously be overstressed without causing immediate failure. The initial yield criterion may provide a rather conservative estimate of the buckling load.

Practical Procedures for Prediction of Strength.—To the writer's knowledge there have been no precise theoretical studies based on the von Karman theory of the strength of beam-columns which fail by bending about both principal axes without twisting. Also, little test data is available. Therefore, practical procedures for the prediction of strength for such cases are based on knowledge of plane buckling. The limiting stress criterion can be used to predict the

buckling load, although this procedure is, in general, even more conservative for cases of buckling about both axes than it is for plane buckling. Ultimate strength formulas also appear applicable.

When the limiting stress criterion is used, the determination of the stresses due to bending about the two principal axes can be made independently as there is no coupling of the flexural actions in the elastic range. Thus, the secant formula, or the solution given by Young (6), or the nomogram by Julian (26) can be extended to include bi-axial buckling.

As in the case of plane buckling, it is necessary to make some allowance for imperfections, which can be done by assuming the existence of an equivalent initial eccentricity or by considering that the stiffness of the column is reduced when the average compressive stress exceeds the proportional limit. That is, the stiffness of the material is assumed to be equal to the tangent modulus E_t, which is a function of the average compressive stress. The same relationship between the tangent modulus and the average compressive stress can be used for this case as is used for plane buckling.

For ASTM A-7 steel the relationship given in Eqs. 1 is appropriate. When the equivalent initial eccentricity procedure is adopted it is customary to assume the existence of an initial eccentricity only in the weak direction. The value of the equivalent initial eccentricity ratio, $e_o\, c/r_y^2$ may be assumed equal to the value used for plane buckling. For ASTM A-7 steel, the values given in Table 1 are appropriate.

The interaction equations for plane buckling can be modified to take account of bending about both principal axes. Thus, Eq. 4 becomes

$$\frac{\sigma_a}{\sigma_c} + \frac{\sigma_{bx}}{\sigma_y\left(1 - \dfrac{0.65\,\sigma_a}{\sigma_{ex}}\right)} + \frac{\sigma_{by}}{\sigma_y\left(1 - \dfrac{0.65\,\sigma_a}{\sigma_{ey}}\right)} = 1 \quad \ldots\ldots \quad (18)$$

and Eq. 13 becomes

$$\frac{\sigma_a}{\sigma_c} + \frac{\sigma_{bx}}{\Gamma_x\,\sigma_y\left(1 - \dfrac{\sigma_a}{\sigma_{ex}}\right)} + \frac{\sigma_{by}}{\Gamma_y\,\sigma_y\left(1 - \dfrac{\sigma_a}{\sigma_{ey}}\right)} = 1 \quad \ldots\ldots \quad (19)$$

in which $\sigma_a = P/A$ denotes the average compressive stress, σ_{bx} and σ_{by} are the maximum flexural compressive stresses (nominal) for bending about the x and y axes, respectively, σ_y represents the yield stress, σ_c is the buckling stress for the member when subjected to concentric compression, and σ_{ex} and σ_{ey} are the elastic critical stresses for buckling in the two principal directions when the member is subjected to concentric compression.

$$\sigma_c = \frac{\pi^2\, E_t}{\left(\dfrac{K\,L}{r}\right)_y^2} \quad \ldots\ldots \quad (20)$$

$$\sigma_{ex} = \frac{\pi^2\, E}{\left(\dfrac{K\,L}{r}\right)_x^2} \quad \ldots\ldots \quad (21)$$

and

$$\sigma_{ey} = \frac{\pi^2 E}{\left(\frac{K L}{r}\right)_y^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

in which $\left(\frac{K L}{r}\right)_x$ and $\left(\frac{K L}{r}\right)_y$ are the effective slenderness ratios for the two principal directions, the former being smaller than the latter, E denotes Young's modulus, and E_t represents the tangent modulus corresponding to the buckling stress σ_c. Note that the buckling stress σ_c corresponds to buckling in the weak direction.

With Eqs. 18 or 19 cases of transverse loading, unequal end moments, and supports other than simple supports can be considered in precisely the same way as explained for plane buckling, since the second and third terms of the interaction formula are independent, and the factors for each can be evaluated as for a plane buckling problem.

COLUMNS WHICH FAIL BY COMBINED BENDING AND TWISTING

Structural members of thin-walled open cross section when subjected to combined axial and bending forces may buckle by combined bending and twisting. This phenomenon is called torsional-flexural buckling or buckling by torsion-bending. The twisting action is a result of the low torsional stiffness of members with open cross section. Members with closed sections, such as box sections, do not twist during buckling to any appreciable extent. It is assumed that these open section members are not subjected to directly applied torsional couples, such as arise when the line of action of transverse loading does not pass through the shear center axis in the undeflection position. The design of members subjected to torsional as well as axial and bending forces is beyond the scope of this paper.

Columns of open cross section will fail by combined bending and twisting under any of the following three conditions:

1. Axial compression and moments acting to cause bending about both principal axes;
2. Axial compression and moments acting to cause bending about the major principal axis when the moment of inertia about the major axis is much greater than about the minor axis; for example, an I-section member with axial compression and end moments acting to cause bending in the plane of the web; and
3. Axial compression and moments acting to cause bending in a direction parallel to either principal axis, when this axis does not contain the shear center as well as the centroid, as may occur for a channel, tee, or an angle section.

In the first and third cases, combined twisting and bending about both principal axes begin at the lowest loads and increase continuously with increasing load until failure by plastic instability occurs. The second case represents a true "eigenvalue" problem in which, for an ideal member, bending occurs in the plane of the loading without twisting as the load increases until the critical load is reached, and then twisting and deflections out of the plane of the loading develop suddenly and simultaneously to cause failure.

H. Wagner (2) and R. Kappus (7) were the first to investigate the torsional-flexural buckling of open, thin-walled cross-section members. Extensive contributions to the theory have been made by J. N. Goodier (8), (10) and Timoshenko (13). Bleich (21) has presented a review of the state of knowledge of this subject. It is fair to say that little information has been obtained on the strength and behavior of members subjected to the first and third conditions. The greatest amount of study has been devoted to the second case and most of this is for I-section members.

I-Section Columns Bending in Plane of Web.—Consider the behavior of an I-section column subjected to combined axial loads and end moments or transverse forces acting in the plane of the web. Such a column fails by a buckling action involving simultaneous twisting and bending in the direction perpendicular to the plane of the web. This behavior is similar to the lateral buckling action of an I-beam subjected to transverse forces only.

On the assumption that the member is initially straight, the elastic critical load may be found in a straightforward manner by a typical characteristic value analysis. The elastic critical load is an upper bound to the collapse load. If the extreme fiber stresses corresponding to the critical load are less than the proportional limit, then the critical load is approximately equal to the collapse load. If the nominal extreme fiber stresses corresponding to the critical load exceed the proportional limit some type of inelastic analysis must be made to find the collapse load, which is, of course, less than the elastic critical load.

Elastic buckling theory has been verified for bi-symmetrical I-sections and eccentric loading by the extensive tests of Hill and Clark (17), (18) on aluminum beam-columns and by the tests of Massonnet (40) on steel columns. Hill and Clark (18) have also shown that a close estimate of the collapse load for I-section members which fail at maximum nominal compressive stresses which exceed the proportional limit (that is, in the inelastic range) can be obtained with the theoretical formula for critical load by the use of the tangent modulus concept.

Johnston (36) has shown for ASTM A-7 structural steel that the relationship between elastic and inelastic lateral buckling strength of beams, determined by tests, is so like the corresponding curve for concentrically loaded columns that a single compromise tangent modulus-stress curve may be used to predict, with good accuracy, the collapse strength of both concentrically loaded columns and beams, and hence, presumably, beam-columns. Thus, the design of columns, beams, and beam-columns may be put on a rational and unified basis for this important material. Essentially the same procedure is recommended for structural aluminum by Hill, Hartman, and Clark (32).

Campus and Massonnet (35) have extended the Wagner-Kappus theory of elastic buckling by torsion-bending to the plastic domain. The results of this extension were compared with results of their tests of steel I-section columns and found to be in excellent agreement.

Consider next some of the important solutions. The buckling of an I-beam by combined axial load and moments in the plane of the web was first considered by Johnston (9) who found the critical load of a simply supported I-beam compressed by an axial load applied with equal eccentricities at both ends. Improved formulas for this case were developed by Goodier (8), (10) Hill, and Clark (18).

Mario G. Salvadori, F. ASCE, (31) has found the elastic critical loads of obliquely loaded I-columns whose ends are free to rotate in the plane of the

web but are either hinged, elastically restrained, or totally fixed in the plane of the flanges at the ends, while they are prevented from twisting around the column axis there. The consideration of unequal end moments in the plane of the web, which may be caused by continuity, allows the use of these solutions for columns that are elastically restrained in both the weak and the strong planes. The solutions are presented in the form of interaction curves.

H. Nylander (15) has found the critical load for mono-symmetrical I-columns (unequal top and bottom flanges) and Tee columns subjected to eccentric loading. The ends are simply supported.

A completely different approach to the study of the strength of I-beams subjected to bending moments and transverse loads acting in the plane of the web is to assume that some accidental initial imperfections exist, to find the stresses which result as the loads are applied, and then to postulate that failure occurs when the maximum fiber stress reaches a specified limiting value. Along the lines of this approach, J. Zickel (19) and Bruno Thurlimann, M. ASCE, (23) have studied the stresses which arise in initially twisted beams, columns, and beam-columns. The solutions of this problem are complex and neither Zickel nor Thurlimann presented simplified equations which might be useful in design. Nylander (15) has derived a practical design procedure based on a simple solution for the maximum stresses in a beam with initial deflections. His procedure may be applied to columns with oblique loading as well as eccentric loading. Monosymmetrical and bisymmetrical I-sections and Tee sections are considered. Finally, Horne (34) has presented a complete design procedure based on the limiting stress concept. Obviously the well-known secant formula is not applicable to this problem, except possibly to give an approximate answer, because no consideration is given to the stresses caused by the twisting.

Practical Procedures for Prediction of Strength of Doubly Symmetric I-Section Members Subjected to Primary Bending in Plane of Web.—Exact formulas for critical loads for torsional-flexural buckling are rather complex. However, the following simple interaction equation has been found to agree closely with available data from tests of simply supported I-section beam-columns subjected to axial compression and bending in the plane of the web:

$$\frac{\sigma_a}{\sigma_c} + \frac{\sigma_{bx}}{\sigma_{cb}\left(1 - \frac{\sigma_a}{\sigma_{ex}}\right)} = 1 \quad \ldots\ldots\ldots\ldots \quad (23)$$

in which $\sigma_a = P/A$ is the average compressive stress at failure, $\sigma_{bx} = M/S_x$ denotes the nominal extreme fiber stress due to applied moments at failure, σ_c represents the average stress at plane buckling in the direction normal to the web for the member subjected to axial compression only, as given by Eq. 20, σ_{ex} is the elastic critical stress for buckling in the plane of the web, given by Eq. 21, and σ_{cb} is the nominal extreme fiber stress at lateral buckling collapse for the member subjected to uniform moment acting to cause bending in the plane of the web. The value for σ_{cb} may be obtained from

$$\sigma_{cb} = \frac{\pi^2 E_t I_y h}{2 S_x L^2} \sqrt{1 + \frac{J G L^2}{\pi^2 E C_w}} \quad \ldots\ldots\ldots \quad (24)$$

in which h is the distance between centers of flanges, J denotes the uniform torsion constant, G represents the elastic shear modulus, and C_w is the torsion

warping constant. Values of the torsion constants for common steel sections are given in a published booklet (16). A complete study of the evaluation of σ_{cb} is given by Clark and Hill (42) and by the CRC (45). In Eq. 24 the tangent modulus E_t should be the value which corresponds to the stress σ_{cb}. Note that the values of the moduli G and E inside the radical are the elastic values. This is essentially the procedure recommended by Hill, Hartman, and Clark (32). Note also that the maximum value of σ_{cb} for short, torsionally stiff, steel beams is the yield strength.

Hill, Hartman, and Clark (32) have found that Eq. 23 gives predictions which agree well with the results of their tests of aluminum beam-columns in both the elastic and inelastic ranges. The members were tested with eccentric compressive load. They also show that when the factor $\left(1 - \sigma_a/\sigma_{ex}\right)$ in the second term is neglected that the formula is unconservative.

Campus and Massonnet (40) also found excellent correlation between the predictions of Eq. 23 and the results of their tests of steel I-section columns under eccentric loading. They also tested members with oblique eccentric loading, moment at one end only and equal and opposite end moments. They found that the results of these oblique eccentric load tests also agreed well with the predictions of Eq. 23 when the equivalent moment procedure previously presented was used with Massonnet's formula. That is

$$\sigma_{bx} = \frac{\left(M_{eq}\right)_x}{S_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

in which $\left(M_{eq}\right)_x/\left(M_1\right)_x$ is determined from Massonnet's formula shown in Fig. 4(b). The possibility of failure at the ends also must be considered as presented in the section on plane buckling.

In a theoretical study, Salvadori (31) found that Eq. 23 along with the equivalent moment procedure gives safe predictions of the combinations of axial compression and bending which produce elastic buckling by torsion-bending. Salvadori considered members whose ends are free to rotate in the plane of the web, but which are elastically restrained with respect to rotation in the plane of the flanges. He found that the effect of this restraint could be accurately accounted for by replacing the total length L in the expressions for σ_c, σ_{ex}, and σ_{cb} by the effective length $(KL)_y$ for plane buckling in the weak direction.

Generalization of Strength Formula.—Massonnet (40) states that Eq. 23 can be used to predict the buckling combinations for design use for every circumstance, for straight prismatic members subjected to simultaneous compression and bending, provided that the proper effective lengths, equivalent moments, and appropriate expressions for σ_{cb} are adopted. The formula is not limited to doubly symmetric I-shaped sections. Possibly Eq. 23 could also be extended to include cases of bending about both axes by adding another term as was done for bi-axial buckling without twisting. Such extensions seem intuitively reasonable. However, to the writer's knowledge there are no data available to verify such intuitive extensions.

DESIGN FORMULAS

It has been shown that the interaction strength formulas are essentially of the same form for beam-columns which twist in failure and for those which do

not twist. Hence, a unified correct basis can be provided for the design of all types of members by design formulas based on the interaction strength formulas. In addition to the advantage of unity, interaction design formulas are simple and convenient to use and by their nature lead to allowable stresses which vary gradually and continuously in a smooth transition from concentrically-loaded columns to beam-columns to beams. Such formulas may include allowances in addition to those previously studied for the imperfections introduced in the normal conditions of construction.

ACKNOWLEDGMENTS

This paper evolved from a report prepared to assist Johnston in the task of writing the Column Research Council Guide (45). The writer wishes to express gratitude for the stimulus provided by the activities of the CRC and for the many helpful discussions he had with the members of the CRC.

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DISCUSSION

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DISTRIBUTIONS OF EXTREME WINDS IN THE UNITED STATES[a]

Closure by H. C. S. Thom

H. C. S. THOM.[13]—The discussion by Lieblein of the application of order statistics to the fitting of extreme wind distributions nicely supplements the paper under discussion. Because the discussion was written, plans are being made to extend his tables to higher subsample values. The writer did not include more of such material for fear of diverging too far from the engineering aspects. Furthermore, the basic theoretical discussion of the distribution was given elsewhere.[2]

The question of efficiency is, of course, important and in fact was uppermost in adopting the method of fitting, but it did not seem appropriate to go into the details in this paper. Certainly the main interest is in the sampling errors in the quantiles; however, there is nothing to show that the quantiles obtained from the method of maximum likelihood are appreciably less efficient than those obtained from order statistics.

Lieblein's approximation to the return period is useful, and the reference to the probability of the design wind occurring during the recurrence interval is interesting.

The writer did not make reference to binomial tables as they were not used. In addition to those Lieblein mentioned, there is Romig's table[14] that covers n = 50 to 100.

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[a] April 1960, by H. C. S. Thom (Proc. Paper 2433).

[13] Chf. Climatologist, Office of Climatology, U. S. Weather Bur., Washington, D. C.

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DESIGN OF WELDED ALUMINUM STRUCTURES[a]

Discussion by Eduard Luss

EDUARD LUSS.[9]—In 1956 the aluminum industry presented the Specifications for Riveted Design of Aluminum Structures. This same source has now (1960) furnished a set of design rules or specifications for welded aluminum structures. These specifications are intended for all types of structures, including bridges. In general, they refer to the shape and sections that the aluminum industry is making available to the designers and represent the test results on these sections.

These design rules are very timely and should dispel the prevailing opinion that aluminum cannot be welded. Actually, welding of aluminum is the only way to get a reliable structure that can survive without painting and maintenance. The best way to show the definite advantage of welded aluminum design is to enumerate the problems of riveted structures.

In reading the specifications and the other material the industry is furnishing to designers, one cannot escape the impression that you are actually reading about steel with an E - value of E_{al}. The industry shows a "steel approach" to this structural material and to the whole problem. From the standpoint of performance and reliability, one difference between these structural materials is in their response to the influence of environment, climate, and other specific conditions of the location. Aluminum and steel alloys are essentially liquid solutions and behave much like water. These solutions are arranged in respect to their plastic range on a temperature scale shown in Fig. 16.

One can see that the physical appearance and qualities such as hardness, inertness, response to heat, and other environmental conditions for these "liquids" vary in the direct proportion to their position on this scale.

According to Z. Jeffries and R. S. Archer,[10] the degree of activities in the metal depends on the inertia of the alloys. Whenever the metal is rendered sufficiently mobile by means of heat, grain growth and internal activity follows rather rapidly.

In other words, the alloys can be compared to water in which a small amount of heat will cause turbulence and convection currents. The same amount of heat will cause less disturbance in aluminum and practically none in steel. Because steel is so high on this scale, one can forget the influence of heat and treat the metal as rigid.

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[a] June 1960, by H. N. Hill, J. W. Clark, and R. J. Brungraber (Proc. Paper 2528).

[9] Bridge Design Engr., Howard-Needles-Tammen and Bergendoff, Cons. Engrs., Kansas City, Mo.

[10] "The Science of Metals," by Zay Jeffries and R. S. Archer, McGraw-Hill Book Co., Inc., New York, 1924.

Temperature cannot be neglected in dealing with aluminum. Certain changes are bound to occur and the internal equilibrium may be upset. Fig. 16 also shows the definite advantage of using this structural metal in the locations with lower year-around temperature.

Aluminum has good protection against atmospheric corrosion. However, the crevises and indentations caused by overdriven rivet heads, gaps around the bolt or rivet holes, micro-fissures at punch holes and scratches, will inevitably result in electrolithic corrosion when any moisture is present. Even if the

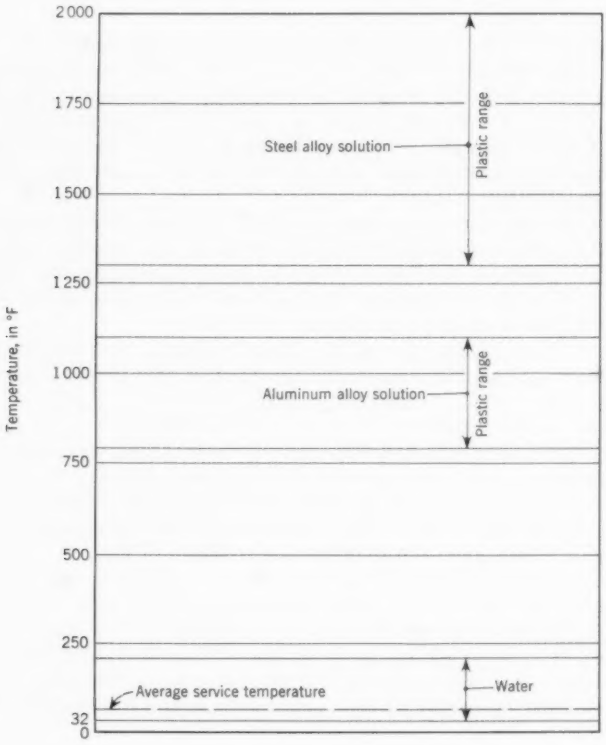


FIG. 16

riveter is extremely careful, the stressed rivet or bolt in contact with an unstressed part, different alloys or metals in contact, aluminum and concrete in contact, can start electrolithic corrosion.

That is, all forms of corrosion such as atmospheric electrolithic, contact, and stress which affect the aluminum alloy and will produce decay in a shorter time, simply because the alloy is more mobile. The only unknown is the time element. How long?

The industry admits that it does not have enough experience. What is known about corrosion from the aircraft industry is not directly applicable to bridges because the life span of a bridge must be about 50 yr, whereas in the aircraft industry only 5 yr to 6 yr are required.

The writer has had the opportunity to examine some aluminum railing that was badly corroded after only about 6 yr of service. There were signs of the anodic stress corrosion at the connections. The bolts had been eaten away, nuts had split and rivet heads had "popped" off. The aluminum was stained with a sediment powder and the junction with the concrete was in poor condition. If these conditions are projected to 50 yr, it is difficult to justify riveted aluminum.

However, one can stop some corrosion by sealing, insolating, and painting. To stop stress corrosion is not as simple, if the detailing is poor.

The writer thinks that one cannot afford to paint aluminum bridges. The good performance of the Canadian bridge in Arvida is due to the special conditions of its location. The temperature is very constant (about 30° F all year long). The designers carefully avoided conditions promoting stress corrosion and used rivets virtually without heads (Fig. 17). It is not always proper to think that any connection which is good for steel can be used in aluminum.

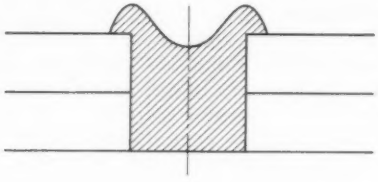


FIG. 17

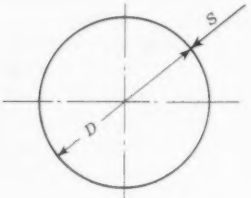
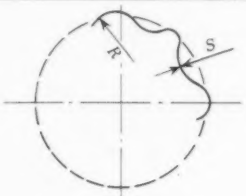
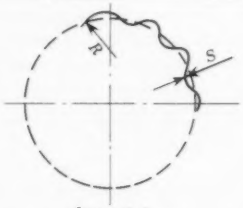
Fatigue tests performed by K. O. Lundberg and G. G. E. Wallgren on some good connections for steel using aluminum, have shown a drop in strength to less than 10% of the static strength.[11] The practical difficulty of using rivets greater than 5/8 in. in diameter increases the number and consequently the points at which corrosion can get started. The bigger rivets require much more pressure to drive and so the chances of harming the parent metal are increased. This is probably one of the reasons for the 5/8 in. rivets used on the bridges under construction in New York state.

The writer thinks that under certain very unfavorable conditions and in conjunction with poor details, the service life of such riveted aluminum structures can become very short indeed, especially if any reliance is placed on the interaction with the concrete deck. The eastern part of the United States has experienced the disintegration of the concrete decks on some bridges. It is clear that, if interaction with the concrete deck is avoided and welding is used, all these heretofore described problems become insignificant, if not non-existant. The structure then has the optimum conditions for service.

11 "A Study of Some Factors Affecting the Fatigue Life of Airkraft Parts with Application to Structural Elements," by K. O. Lundberg and G. G. E. Wallgren, Flugtekniska Forseksanstalten FFA Medd. 30, Stockholm, 1949.

In welding only a definite drop of strength within the narrow welding zone, that is well defined in the design rules, must be accounted for. Bridge designers can always find a place in which this drop of strength will not harm the overall design. It is also known that there are some methods used in the missile industry, that allow welding of aluminum with practically no loss in strength. At present these methods are too expensive for bridges, but this may be dif-

TABLE 3

Column section	Buckling load, in kg	Percentage
 $\frac{D}{S}=166.5$	4 650	100
As above, only with transverse stiffeners	5 500	118
 $\frac{R}{S}=58.3$ Corrugated	5 420	117
 $\frac{R}{S}=35$ Corrugated	7 750	167

ferent in the future. The industry is developing new weldable alloys that regain the lost strength after a period of time. All this is very encouraging and will help aluminum become suitable bridge building material.

What can be done with the shapes and the research provided by the industry so far, is demonstrated by the few aluminum bridges already built. These bridges were designed by such outstanding designers as N. Ashton, who design-

ed the first all-welded continuous plate-girder bridge (Iowa bridge), and George W. Zalite, M. ASCE and E. I. Ziegler, M. ASCE who designed several riveted plate-girder bridges now (1961) under construction in New York state. In both cases the designers had to resort to composite design action to keep the amount of metal and cost down.

The aluminum industry presented another type of bridge, the Fairchild bridge. This bridge is said to be very economical and represents an attempt to carry moments on the shear section. The design oversimplified the action (Bredt formula) as well as the interaction with the concrete deck and is not the most economical. The writer doubts the adequate performance of this structure when projected to the 50 yr of required service.

These three bridges represent the utmost that can be done in this field using the familar steel shapes and "the steel approach." At the same time it represents all that the aluminum industry is furnishing the bridge designer.

The writer is of the opinion that the problem of simple structures in aluminum, as applied to bridges, is largely a problem of deflections. To make it a problem of strength the designer has to go to statically indeterminate structures and quite different structural shapes and connections. The problem of shape or form is the most important factor in low modulus metals and it has been grossly overlooked.

Cylindrical columns tested in Germany in 1956,[12] show the remarkable increase in buckling load obtained by changing the shape of the section; Table 3 shows the results. It is seen that a plain circular section, if reinforced by transverse stiffeners, will carry 18% more load. With various corrugations the same section will carry from 17% to 67% more.

The big increase in the capacity of columns, if they are corrugated, was reported by Kreissig.[13] The shapes can be extruded in one operation. Such columns, in combination with high yield point aluminum alloy, can certainly show very impressive savings in metal and money.

In general, the writer thinks that the designer should be given the choice of the most suitable alloy for the required stress. The alloys suggested in the design rules will not result in the most economical compression members. In the compression members the effect of welding can lead to an unnecessary increased section. Why not try to develop some sort of sleeve connection for this purpose?

Perhaps it would be possible to coldwork the heat-affected zones. This method was used in 1947 in France by G. Ferrand.[14] The strength of welded high strength steel pipe was increased by stretching the diameter about 5%. The coldworked pipe, stretched over the proportionality limit, showed excellent qualities. This would be feasible for closed aluminum sections.

Today, one can design structural shapes that will even eliminate the need to weld. This is possible because there are available excellent epoxy resin adhesives that easily develop more than 13,000 psi in shear. The writer has developed such a girder with no welding at all and it is shown in Fig. 18. The box section is a combination of two identical parts; the upper part has a groove, the lower part a tongue. Before both parts are locked together, the necessary

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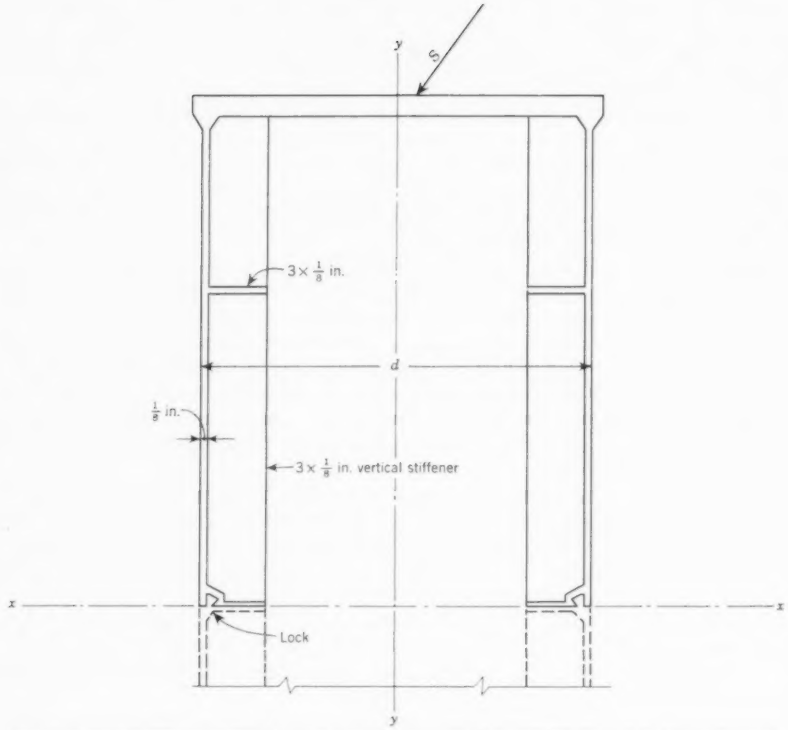
[12] Tiefbohrmast in Rohrschalenbauweise, Der Stahlbau, E. Bahke, July, 1956.

[13] Mitteilungen der Blechforschung, Kreissig, December, 1951.

[14] "Structures de la conduite forcee unique pour hautes chutes a grand puissance," by G. Ferrand, Memoires, 1947.



Section	$H/2$	d	S	Area
A	24 in.	18 in.	$1\frac{1}{2}$ in. by 18 in.	72 sq in.
B	16 in.	14 in.	$\frac{1}{2}$ in. by 16 in.	30 sq in.
C	16 in.	14 in.	$\frac{5}{16}$ in. by 16 in. (corrugated)	26 sq in.

FIG. 18

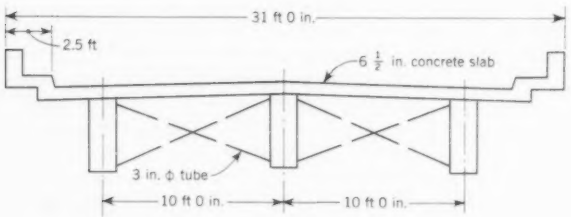


FIG. 19

transverse stiffeners are glued into place as required by stress. After that, the two parts are joined and the girder is ready. The horizontal shear is taken in the key and on the horizontal leg of the longitudinal rib by adhesives.

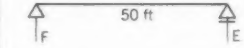
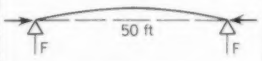
In Fig. 19 there is shown a section of a concrete deck bridge. The span is 50 ft and the action is non-composite.

The total amount of metal is shown in Table 4. It is computed for three different statical types A, B, and C:

Type A is a simple beam on which the section is governed by the live-load deflection. The height of the girder is 4 ft and the required extrustion is with a 30 in. circle. This is the maximum possible extrusion at this time (1960).

Type B is a beam with one end fixed. This is a problem of strength and the design required the combination of two alloys. The suggested yield points of these alloys should be as designated in the notes or better. The height of the

TABLE 4

Bridge type	Structural system	Governing case	Girder section type	Total metal, in lb per sq ft	Notes and suggested alloy
A		Deflection	A	9.00	6061-T6
B		Stress	B	4.50	6061-T6 better than 7176-T6
C		Stress	C	3.50	Better than 7176-T6
Fairchild Bridge	Shear cells $H \cong 4.8$ ft span 50 ft	—	Cellular composite	9.40	Experimental bridge 6061-T6
New York State bridges	Riveted plate girders, long spans	—	Composite	6.25 9.60	6061-T6

box is 2 ft 8 in., and the extrusion can be made by any producer. The metal savings are considerable but the fixed-end connection could be a weak spot if not properly detailed.

Type C is a beam with both ends hinged. The temperature stresses are taken by the section. The girders are cambered and the shoes are fixed in such a position that under loads the maximum stresses will be compressive. The effect of so introduced prestressing and arching makes the structure a stress problem. The section of the girder is, as for Type B, with the top and bottom plates being corrugated. High yield point alloys are required but the shear problem is not too important.

For comparison, Table 4 also shows the amount of metal used in the bridge now under construction in New York. Table 4 demonstrates the advantage of using statical indeterminancy rather than the more conventional way of increas-

ing the stiffness by using deeper sections. The latter has its limitations and is not as economical. This economical disadvantage becomes more pronounced for medium size spans. The answer here appears to be in the development of high yield strength alloys to take compression and special new structural extruded sections.

Conclusions.—The new welding design rules are of great importance and value because a welded bridge offers the minimum opportunity for any type of corrosion, that, in itself, is a precondition for the required long service life. A structure must be designed to last at least 50 yr without heavy maintenance or painting.

However, from the standpoint of bridge design, the importance of alloys and new shapes should be realized, and research in this direction intensified. This will help to solve the deflection problem. Other means of joining the members, such as the new adhesives and sleeves, should be explored. Painting of an aluminum structure should not be regarded as normal practice. A critical evaluation of steel shapes and sections should be conducted to determine their feasibility in aluminum.

The writer thinks that if the industry has a serious desire to further the use of aluminum in bridges, an entirely new approach, as outlined, should be attempted. The industry must admit that, to date, available research and shapes cannot satisfy the specific requirements of bridge designers and that, consequently, the presented design rules have only limited value for this special field.

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23. Corrosion-Causes and Prevention, Speller, 1935.

CHARTS FOR DESIGN OF REINFORCED CONCRETE COLUMNS[a]

Closure by W. H. Gardner, Jr., and Donald H. Kline

W. H. GARDNER, JR.,[16] A. M. ASCE, and D. H. KLINE,[17] A. M. ASCE.—In his discussion Mr. Krusling correctly and concisely explained a modified use of the charts for rectangular columns. Actually, not only the width, but also the depth, may be varied, although this latter operation is a little more involved and probably not worthwhile except when designing a number of columns of the same size.

Mr. Begg suggested several ways in which the charts could be more directly applied to design problems, principally the plotting of moment instead of eccentricity and actual bar combinations instead of percentages of reinforcing. The parameters used were chosen to permit the drawing of one set of base curves for each concrete strength. Different scales were drawn on transparencies of these base curves, resulting in the charts presented. Constructing the charts in this simplified manner would not have been possible if moments and bar combinations had been plotted.

TABLE 2.—REINFORCING FOR ONE-WAY BENDING; PERCENTAGES OF SQUARE COLUMN GROSS AREA, P_t

Reinforcement	Side, in Inches					
	12	14	15	16	18	20
Four 14S		4.60	4.00	3.52	2.78	2.25
Four 18S					4.94	4.00
Six 14S		6.89	6.00	5.27	4.17	3.37
Six 18S						6.00

Note.—This table includes as many bars for each section as will fit along two faces of the column in accordance with ACI 318 - 56, 1103 (b) and 1104 (a). Reinforcement placed in this manner is in agreement with assumption made in constructing ultimate load charts.

The writers wish to note that the column design tables mentioned as being computed are available.[18] These tables are arranged and indexed substantially along the line suggested by Mr. Begg and are extremely useful.

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[a] May 1960, by W. H. Gardner, Jr. and Donald H. Kline (Proc. Paper 2461).

[16] Assoc. Prof. Civ. Engrg., Duke Univ. & Partner, W. H. Gardner, Jr. & Assoc., Durham, N. C.

[17] Partner, W. H. Gardner, Jr. & Assoc., Durham, N. C.

[18] "Reinforced Concrete Column Tables Ultimate Strength Design," by Hugh Fenlon, Atlanta, Ga.

Table 1 presented the percentage of reinforcing for various bar combinations possible with bars #11 and smaller. Because use of 14 S and 18 S bars is becoming more widespread, Table 2 is offered as an extension of Table 1 to include these bars.

LONG SPAN PRESTRESSED CONCRETE FOLDED PLATE ROOFS[a]

Discussion by A. A. Eremin

A. A. EREMIN,[6] M. ASCE.—As noted by the authors, a deficiency in the compressive zone of long span folded concrete roof slabs is often found. In addition to the methods of overcoming this deficiency cited by Brough and Stephens, one may add the increasing of the compressive zone area by gradually increasing the thickness of the slab toward the top of the folds. In so doing, the aesthetic effect of the folds is not destroyed.

As is well known, an expensive element of cost in the construction of this type of structure is the required falsework. In precast and prestressed folded slabs this cost can be reduced by precasting the slab on the ground in units of one or two folds. The completed units may then be lifted and installed in place in the same manner as for bridge girders.

In constructing folded roof slabs, great care must be used to determine the sequence of placing the concrete. It would be of interest to know what placing sequence was used in the illustrative example.

[a] October 1960 by John C. Brough, Jr. and H. Stephens, Jr. (Proc. Paper 2630).
[6] Cons. Structural Engr., Berkeley, Calif.

FLOW GRAPHS IN STRUCTURAL ANALYSIS[a]

Discussion by H. L. Su

H. L. SU.[6]—Congratulations are extended to the author for introducing a novel approach to the solution of problems of structural analysis.

Numerical methods of linear analysis can, in general, be classified as "exact" or "approximate." In the exact methods attention is concentrated on some "rigorous" techiques of mathematical operations. In the approximate, the focus is generally on some simple arithmetical maneuvers that can be easily mastered and can often be interpreted physically or pseudo-physically. The flow graph introduced by Gerstle seems to fall in between these two established poles and stands out as a powerful tool to attack problems of linear analysis.

A similar approach, the amplification method (that is a derivative of topological network analysis) was developed to solve problems of structural stability.[7] Stability problems require a simple and efficient method of analysis due to their non-linear characteristics. This method is based on the fact that any linear conservative system will amplify or attenuate an applied disturbance. The relation between such a disturbance and its induced response is an invariant that has been revealed by the "dependence" in the paper. When this invariant, or rather all the invariants necessary for analysis have been found, solution for problems due to any combination of disturbance can be obtained immediately.

Such a direct approach suggests a new path for the development of skeletal complex analysis that is still dominated by the classical "cut and fit" artifices. The main objection to such techniques is that "cuts" are arbitrarily made purely for the purpose of reducing a structural configuration to an amputated "determinate" state that has little physical correspondence in reality. They are more mathematical then physical in nature. Consequently, these methods of analysis do not normally provide a sufficiently direct and explicit guide to design.

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a October 1960 by K. H. Gerstle (Proc. Paper 2634).

6 Analogue Lab., Imperial College, S. W. 7, London, England.

7 "On Stability of Two-dimensional Structural Frameworks," by H. L. Su, Proceedings, I.C.E., London, Vol. 16, June, 1960, pp. 143-146.

PRINCIPLE OF VIRTUAL WORK IN STRUCTURAL ANALYSIS[a]

Discussion by A. R. Oliver and Alexis Ostapenko

A. R. OLIVER.[21]—It is pleasant to see such a refreshing approach to the virtual work principle. The writer has, for some time, advocated that all the energy theorems be divorced from the physical agrument that usually accompanies them, and be restated as mathematical artifices in this way.

The author is mistaken, however, in stating in his discussion of Eq. 8 that it "could not possibly be derived from any energy principle." In fact, it is quite simply derived in a standard manner from the theorem that states that the total potential energy of the forces in a system is a minimum with respect to their displacements (The word 'forces' is taken to include loads, shears, and moments so long as the corresponding displacements are used.)

Adopting the author's notation with the addition of the symbol ϕ for the curvature and using his definition sketch (Fig. 1) the total potential U of the system is defined by the equation

$$U = \int_A^B \int_0^y W(x)\, dy\, dx + \int_0^{y_B} V_B\, dy_B - \int_0^{y_A} V_A\, dy_A + \int_0^{\theta_A} M_A\, d\theta_A$$

$$- \int_0^{\theta_B} M_B\, d\theta_B + \int_A^B \int_0^{\theta} M\, d\theta\, dx \qquad \ldots\ldots\ldots\ldots (67)$$

No information is, as yet, available to define the functional relationship between each force and its corresponding displacement, so that all of the variables are, at this stage, separate and independent. However, it is assumed that such a relationship can eventually be found so that the integrals can be evaluated if desired. The theorem of minimum potential energy then states that U is a minimum with respect to variations of y subject to the following equations of geometry:

$$\phi = y'' \qquad \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

and

$$\theta = y' \qquad \ldots\ldots\ldots\ldots\ldots\ldots (69)$$

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[a] November 1960 by Frank L. DiMaggio (Proc. Paper 2643).

[21] Prof. of Civ. Engrg., Univ. of Tasmania, Hobart, Tasmania.

Eqs. 68 and 69 were not specifically stated by the author, but were used by him. Eq. 68 is required for all points on the beam, but Eq. 69 is needed only at ends A and B.

The variations are conveniently applied in the manner adopted in the calculus of variations.[22] We suppose that $y = y(x)$ is the function that makes U a minimum, then for all functions in the immediate neighborhood of y, U is least when $y = y(x)$. Consider a function $\eta = \eta(x)$, any arbitrary function of x, continuous and with its first and second derivatives continuous over the length from A to B. (A finite number of discontinuities is permitted if each is treated in the manner adopted by the author. This function η has exactly the same significance as the author's function with the same symbol. A variation δy is obtained by putting

$$y + \delta y = y + \alpha \eta \qquad \ldots\ldots\ldots\ldots\ldots \quad (70)$$

in which α is a parameter that may be made arbitrarily small, as in the ordinary process of going to a limit. If U is to be the required minimum, the function $U + \delta U$ obtained by putting $y + \delta y$ for y in Eq. 67 is a minimum for $\alpha = 0$ when considered as a function of α, and this must hold for any arbitrary function η. The substitution yields

$$\delta U = \alpha \left[\int_A^B W\eta \, dx + V_B \eta_B - V_A \eta_A + M_A \eta'(B) + \int_A^B M \eta'' \, dx \right] \ldots \quad (71)$$

that, on differentiating with respect to α and equating to zero gives

$$\int_A^B W \eta \, dx + V_B \eta_B - V_A \eta_A + M_A \eta'(A) - M_B \eta'(B) = - \int_A^B M \eta'' \, dx \ldots \quad (72)$$

Except for the term introduced by the author to permit a discontinuity in $\eta'(x)$ at $x = x_S$, Eq. 72 is the same as Eq. 8, so that the principle of virtual work is the same as the theorem of potential energy. It is often convenient to use the principle in the way indicated by the author, but it is frequently more useful to define a partial derivative so that the same result is obtained from the equation

$$\frac{\partial U}{\partial y} = 0 \qquad \ldots\ldots\ldots\ldots\ldots \quad (73)$$

Eq. 73 is particularly valuable in approximate methods of analysis where an assumed shape is used for $y = y(x)$ with parameters adjusted to satisfy this equation. Both Eqs. 72 and 73 are independent of the stress-strain law for the material of the beam so long as it is single-valued or a loading path can be defined.

It must be emphasised, however, that the foregoing proof of identity of the two principles is not a proof of validity of either. The writer has pointed out[23]

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[22] "Methods of Mathematical Physics," by R. Courant and D. Hilbert, Interscience, Vol. I, 1953, p. 184.

[23] "Energy Methods in Structural Analysis," by A.R. Oliver, Transactions, I.E. Aust. Vol. CE3, No. 1, March, 1961.

that the latter proof can be obtained only by examining the conditions that Eq. 72 is valid for any arbitrary function η, that is simply done by tracing backwards the path followed by the author. The term on the right hand side is integrated by parts to give

$$\int_A^B M\,\eta''\,dx = M_B\,\eta'(B) - M_A\,\eta'(A) - \int_A^B M'\,\eta'\,dx \quad \ldots\ldots \quad (74)$$

and the integral on the right, again integrated by parts, resulting in

$$\int_A^B M'\,\eta'\,dx = M'(B)\,\eta_B - M'(A)\,\eta_A - \int_A^B M''\,\eta\,dx \quad \ldots\ldots \quad (75)$$

When the results of these integrations are substituted in Eq. 72, it can be seen that the latter is true for all functions η, only if at all points on the beam

$$W + M'' = 0 \quad \ldots\ldots\ldots\ldots \quad (76)$$

and at the ends A and B

$$M' = V \quad \ldots\ldots\ldots\ldots\ldots \quad (77)$$

Eq. 76 is an equation of statical equilibrium identical with Eq. 1 that DiMaggio has applied at all points on the beam and Eq. 77 identical with Eq. 3 that the author has applied at the ends A and B. Because Eqs. 76 and 77 are valid then Eqs. 72 and 73 that replace them are also valid. The significant feature of this development is that it shows that the principle of virtual work and the theorem of minimum potential energy are merely convenient restatements of the equations of statical equilibrium and do not add any new information to the problem.

There is a natural counterpart to these two theorems that results in a theorem of minimum (or maximum) complementary energy and a principle of virtual loads, that ought to be mentioned here to complete the picture.

The complementary energy C of the system of forces used by the author is defined by the equation

$$C = \int_A^B \int_0^W y\,dW\,dx + \int_0^{V_B} y_B\,dV_B - \int_0^{V_A} y_A\,dV_A + \int_0^{M_A} \theta_A\,dM_A$$

$$- \int_0^{M_B} \theta_B\,dM_B + \int_A^B \int_0^M \phi\,dM\,dx \quad \ldots\ldots\ldots \quad (78)$$

The theorem of minimum complementary energy then states that, subject to the restrictions of the equations of statical equilibrium, that is Eqs. 73 and 76, C is extremal with respect to variations of the loads (or bending moments) and this replaces the equations of geometry, that is Eqs. 68 and 69. The variations are applied in the same way as before, and there is chosen a function $\gamma = \gamma(x)$,

an arbitrary function of x of the same sort as η and for convenience of notation the variation is applied to the bending moments rather than to the loads so that a variation δM is obtained from

$$M + \delta M = M + \alpha \gamma \quad \ldots\ldots\ldots\ldots\ldots (79)$$

that leads in the same manner as before to the equation

$$\int_A^B y \gamma'' \, dx + y_B \gamma'(B) - y_A \gamma'(A) + \theta_A \gamma_A - \theta_B \gamma_B + \int_A^B \phi \gamma \, dx = 0 \ldots (80)$$

Eq. 80 is a statement of the principal of virtual loads, although γ is no more a bending moment than η is a displacement. Again, it is frequently more convenient when dealing with approximate processes to define a partial derivative so that

$$\frac{\partial C}{\partial M} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (81)$$

In order to establish the validity of Eqs. 80 and 81 it is necessary to find the conditions for which they will hold for all functions γ, that by the same method used for the minimum potential energy theorem leads to the conclusion that Eq. 68 must hold for all points of the beam, and Eq. 69 at the ends A and B.

Eq. 81 is a restatement of the "least work" theorem in a form that is valid for non-linear stress-strain laws. The usual statement of this theorem replaces the C with U that is valid for linearly elastic materials only. In either case the theorem expressed by this equation simply replaces the equation of geometry and adds no new information to the problem.

There are, thus, two reciprocal theorems, the strain (or potential) energy approach in which the constraints are equations of geometry and the energy equations replace equations of statical equilibrium; and the complementary energy approach, in which the constraints are equations of statical equilibrium and the energy equations replace the equations of geometry. Likewise, there are two ways of building up energy equations, in the case of strain energy, one may start with an equation of statical equilibrium and build up an energy function to replace it as the author does, or start with a likely looking energy equation and establish the equation of statics that it replaces as has been done in this discussion. All of these processes have advantages in different applications and should be all considered together.

ALEXIS OSTAPENKO,[24] M. ASCE.—A novel and convincing derivation of the principle of virtual work presented by the author should be noted by all who use this principle, especially those engaged in teaching. Examples of the application of the principle to materials other than elastic are also very valuable.

Although DiMaggio apparently endeavored to describe the treatment of all the effects causing deformation in engineering structures, the case of a uniform

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[24] Asst. Prof. of Civ. Engrg., Lehigh Univ., Bethlehem, Pa.

change of temperature was somehow overlooked; that is, the case when the temperature may vary along a member but has no gradient across the depth. To include this effect in the deflection computations a term $\alpha\ T_0$ should be added to Eq. 53:

$$u' = g(\bar{T}) + \alpha\ T_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (82)$$

in which α is the coefficient of thermal expansion and T_0 is the uniform change of temperature that is the difference between the temperatures at the given time and at the time of construction. (It is very unfortunate that the author used the letter T to denote both the thrust and the temperature; it would have been clearer if, for example, the letter N had been used to designate the thrust.) Then Eq. 54 becomes

$$y_m(i) = \sum_S T\left[g(\bar{T}) + \alpha\ T_0\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (83)$$

The writer considers it important to state that Eq. 62, derived for the computation of elastic deflections including the effect of shear, is more general and "automatic" than the more familiar Eqs. 64 and 65. Whereas Eq. 62 can be used directly for the computation of the slope in any structure by using an equilibrium system caused by a unit couple at the desired point, Eq. 65 gives no contribution due to shear in statically determined structures. To include the effect of shear on the slope in the latter case it is necessary either to "shift the gears back" to Eq. 56 or, what is conventionally done, to use an artifice in which the required unit couple is applied as a pair of opposite forces each equal $1/c$ at a distance c from each other.

In conclusion, it might be desirable to assemble in one place all the expressions derived for the computation of deflections and slopes of elastic engineering structures:

$$\eta(\bar{x}) \quad (\text{or } \eta'(\bar{x})) = V_A\,\eta(A) - V_B\,\eta(B) - M_A\,\eta'(A) + M_B\,\eta'(B) - T_A\,u(A) + T_B\,u(B)$$

$$+ V(x_d)\ \delta_d - M(x_s)\ \theta_s$$

$$+ \int_S M\left(\frac{\bar{M}}{E\,I} + \frac{k\,\bar{W}}{A\,G} + \alpha\,\frac{\Delta T}{h}\right) dx + \frac{k\,P}{A\,G}\ M(x_p)$$

$$+ \int_S T\left(\frac{\bar{T}}{A\,E} + \alpha\ T_0\right) dx \quad \ldots\ldots\ldots\ldots \quad (84)$$

In Eq. 84 the first line gives the contribution of the movements and rotations of the supports; it is assumed that the equilibrium system has two supports A and B. The second line shows the effect of sudden changes in the geometry of the structure. Strictly speaking, the second line should also include a term $T(x_f)\ \Delta L_f$ that would take care of a sudden change in the length of a member, ΔL_f, at point f (due to lack of fit, for example). The third line covers the deformation caused by bending, shear and temperature gradient, and the fourth line gives the effect of thrust and uniform change of temperature.

CONCEPTS OF STRUCTURAL SAFETY[a]

Discussion by Jack R. Benjamin

JACK R. BENJAMIN,[31] M. ASCE.—This excellent philosophical review of the structural safety problem raises many questions that have long troubled structural engineers. The writer is a confirmed believer in the statistical approach to structural engineering. However, the problem is by no means as simple as one might believe from a surface investigation.

Civil engineering structures differ in one major respect from aircraft structures. This is the matter of responsibility. The structural engineer is individually responsible for the performance of his design, both morally and legally. In view of this paramount responsibility problem, the probability and statistical approach to structural design can never be expected to entirely replace tried and true judgment procedures. Probability and statistical procedures offer two more powerful tools the engineer can bring to bear on his problems. The detailed techniques are far from perfect at present (1961) due to lack of data and almost complete lack of training in probability and statistics on the part of those who attempt to apply to concepts.

The theory of probability has been viewed the key to a more rational derivation of load factors. The concept has been to jump from the theoretical probability distributions directly to application in design with almost no attention being paid to the statistical part of the problem. The determination of the proper distribution and the evaluation of the distribution parameters are statistical problems. The theory of probability is a rigid mathematical discipline. The making of decisions based on probability models involves statistics and a rigid mathematical discipline.

As engineers we must recognize the difference between mathematics and judgment. Judgment decisions cannot be made directly on the theory of probability without consideration of the statistical problems involved. It is this very point that invalidates every load factor currently in use. Fig. 1 might be used to compare code load-factors and the results from theory of probability. The curves do show that the arbitrary code load-factors have a shape similar to the curves derived from mathematical models of the theory of probability. However, the ordinate relationships have absolutely no rationality. The probability models were arbitrarily selected to provide curves in the same region as those sketched for the pure judgment factors. We could juggle coefficients to make a particular code equation fit reasonably well to a particular probability model again with arbitrary parameters. Such a procedure provides just another mathematical curiosity. Statistical theory must be used to bridge the gap between design and probability concepts.

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[a] December 1960 by C. B. Brown (Proc. Paper 2678).

[31] Prof. of Structural Engrg., Stanford Univ., Stanford, Calif.

The application of statistics to design will result in design based on probability of failure. However, we will not be able to say that the probability of failure is a certain value. Rather we must say that the conclusion that it is this value contains α and β errors.

For a simple analogy, if we are making a deflection theory arch analysis, once the mathematical work begins, judgment decisions are eliminated until the results of the analysis are known. We cannot insert arbitrary judgment approximations into the middle of the computations and expect valid results. Similarly, once we commit ourselves to probability analysis, the next and succeeding mathematical steps must be taken or the original statement is of no value. Thus, the load factor procedure for combining loads is not valid. It is pure engineering judgment and must be recognized as such.

The concept that the vote of a committee can establish scientific truth is absurd. If "safety level" is to be decided by a committee, it will be a judgment decision based on majority vote. This is not the direction in which scientific progress can be made. If structural design is to become more rational using the theories of probability and statistics, the decisions must be made on the mathematical results of studies within a known statistical context. After the mathematical results are known, judgment must then be used in their application to particular structures under real loadings.

PERIODS OF BUILDINGS FOR EARTHQUAKE ANALYSIS[a]

Discussion by Clarence J. Derrick

CLARENCE J. DERRICK,[16] F. ASCE.—The authors' direction of attention to influences that affect the value of free period is most commendable. Although their proposed formula follows precedent by use of the ratio, $L/\sqrt{d}$, their Fig. 6 shows the impropriety of relying on this quantity as the sole criterion of free period variation.

An examination of earthquake "spectra" shows that no high degree of accuracy is required to estimate the free periods of actual structures. Moreover, this characteristic is not likely to remain constant throughout the range of probable response distortion. In such circumstances, the necessity for considering, by specific computation, the effects of "translational motion" and "rocking," is debatable. Probably a correction might be made, with sufficient accuracy in actual situations, by an estimated increase of the combined "shear" and "bending (or 'flexural')" free period.

To estimate the contribution to the free period by "flexural" action, the simple assumptions used in the presentation appear sufficient for ordinary situations. The solution is somewhat complicated by the necessity of using tabular coefficients. A reasonably accurate estimate of this contribution may be made more simply.

Because, for estimating the value of free period, the assumed shape of the dynamic distortion may vary considerably without introducing much error, there appears to be no necessity for developing the theoretical attidude in "shear" deflection. This complexity is even less justifiable when the rather simple situation assumptions are considered. The conventional method by "static" loading and "equating energies" is more flexible.

The objection to the conventional method, by "equating energies" (that is that story rigidities must be appraised) loses force when considering the distribution of story shears, regardless of how deduced. The principal advantage of the conventional method lies not in its flexibility of application or accuracy of result but in a "by-produce" - the pattern of "shear" distortion. The authors' expression for "shear" contribution not only does not reveal this pattern but, by complexity, may even obscure it.

The importance of the pattern of "shear" distortion has been enhanced by the modern trend toward consideration of two levels of excitement and two types of response.[17] At the lower level - the "moderate" earthquake - and "elastic" response, the "shear" distortion pattern developed by the conventional "equa-

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[a] December 1960 by Mario G. Salvadori and Ewald Heer (Proc. Paper 2680).

[16] Cons. Engr., Los Angeles, Calif.

[17] "Behavior of Structures During Earthquakes," by G. W. Housner, Proceedings, ASCE, Vol. 85, No. 4, October, 1959, p. 128.

ting energies" method, corrected to the "fundamental" mode shape, reveals, by inspection, the risk of "superficial" damage along the height of the structure. At the upper level - the "maximum" earthquake - and "elasto-plastic" response, this pattern indicates location of "soft spots," stories in which the "elastic" yield varies abnormally. From present studies of "elasto-plastic" performance, the location of such "soft spots" appears usefully indicative.

If excess distortion, developed by "plastic" yielding is equally proportional to "elastic" yielding, in all stories along the height, the overall effect will be a lengthening of free period and an increase in "absorption of energy," over the values obtaining in "elastic" performance. At moderate levels of "elasto-plastic" action, the only permanent effect is additional "superficial" damage.

Response of a building to base motion is an "elastic" phenomenon. A "plastic" structure may be deformed but it will not respond. The "shear wave" moves along the height of a multi-story building by "bounds," from floor to floor. A "soft spot" is an invitation to cessation of such transmission and consequent concentration of displacement. G. N. Bycroft has concluded.[18] "It should be the object of aseismic design to minimize and distribute these yield displacements over the entire structure. The achievement of an optimum design, however, is found to be very complicated. The relative displacements depend more critically on the stiffness and strength configuration than in the completely linear case. Thus, any particular design may be comparatively weak at some point in its height though adequate elsewhere."

At the lower level of excitement - the "moderate" earthquake - minor errors in estimating free period are not important both because of the "spread" between "allowable frame stresses" and "elastic limits" and because risk of "superficial" damage is not accurately definable. It is not the total relative displacement between roof and base that is important but rather the proportion of this displacement developed in each story.

At the upper level of excitement - the "maximum" earthquake - in which the frame is expected to perform "elasto-plastically," the safety of the structure may depend on the "stiffness and strength configuration." The conventional method of estimating free period by "equating energies" seems to be somewhat more flexible and much more informative than the procedure recommended by the authors.

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18 "Yield Displacements in Multistory Aseismic Design," by G. N. Bycroft, Bulletin, Seis. Soc. of America, Vol. 50, No. 3, July, 1960, p. 453.

STATIC AND DYNAMIC ANALYSIS OF GUY CABLES[a]

Discussion by Glen V. Berg

GLEN V. BERG,[8] F. ASCE.—The author's dynamic analysis treats the motion of a guy cable due to a sinusoidal horizontal motion of one of the supports in the plane of the cable. The problem is difficult, and the author has used simplifying assumptions to make it tractable. However, some of the assumptions are questionable and lead to unacceptable results.

In manipulating the equation of motion, Eq. 41, the author uses Eq. 7, an equation for T that was derived from static equilibrium considerations. This implies that Eq. 7 holds regardless of the motion of the cable. Later, in the separation of variables following Eq. 44, H is treated as a constant. While these steps would be valid in the classical vibrating string problem, they can hardly be expected to hold where the initial sag of the cable is not necessarily small, and where longitudinal and transverse motion occur. It is somewhat puzzling that in the solution Eq. 56 gives the time variation in H that was assumed to be constant in the first place.

The equation for the cable motion might, at first glance, appear tenable, for it has some of the characteristics that one would expect. Combining Eq. 48(a) with Eq. 52,

$$\eta^*(\bar{x}) = \sin \gamma_1 \left\{ \frac{\cosh(2\,r + a_1)}{\cosh(2\,r\,\bar{x} + a_1)} \right\}^{1/4} \left\{ \frac{\sin \beta\, D_{(\bar{x})}}{\sin \beta\, D_{(1)}} \right\} \quad \ldots\ldots (61)$$

in which $\eta^*(\bar{x})$ is the ratio of transverse displacement at $\bar{x}$ to horizontal displacement of the support, and other notation is that used by the author. The values of β for which $\sin \beta\, D_{(1)}$ is zero would appear to be associated with the natural frequencies of the cable. But two discrepancies arise:

First, because γ_1 is the angle of inclination of the cable at the movable support in the equilibrium position, there is no reason why it could not be zero. At least Dean has imposed no such restriction. In this case $\eta^*(\bar{x})$ would be zero for all values of $\bar{x}$, and there would be no transverse motion of the cable--an obvious contradiction. Fig. 6 is such a case, computed from the equations given by the author.

Second, consider the case in which the movable support oscillates slowly. From the integral expression for $D_{(\bar{x})}$ in Eq. 49

$$0 \leq D_{(\bar{x})} \leq D_{(1)} < \sqrt{\cosh(2\,r + |a_1|)} \quad \ldots\ldots\ldots\ldots (62)$$

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[a] January 1961, by Donald L. Dean (Proc. Paper 2703).

[8] Assoc. Prof. of Civ. Engrg., Univ. of Michigan, Ann Arbor, Mich.

Q = 10.55 LBS/FT
H = 20.00 KIPS
HOR SPAN = S = 950.00 FT
HEIGHT = -243.06 FT
PERIOD = 5.00 SEC

ALPHA = 1.2566 RADIANS/SEC
A1 = -0.5011 A2/S = -2.2514
GAMMA 0 = -0.4814 GAMMA 1 = 0.0000
R = 0.2506 BETA = 4.8341

X/S	Y/S (EQ. 4)	T/H (EQ. 7)	D (EQ. 49 INTEG)	ETA* (EQ. 61)
0.00	0.000	1.128	0.000	0.000
0.10	-0.049	1.103	0.106	0.000
0.20	-0.093	1.081	0.210	0.000
0.30	-0.132	1.062	0.314	0.000
0.40	-0.165	1.046	0.416	0.000
0.50	-0.193	1.032	0.518	0.000
0.60	-0.216	1.020	0.619	0.000
0.70	-0.233	1.011	0.720	0.000
0.80	-0.246	1.005	0.821	0.000
0.90	-0.253	1.001	0.921	0.000
1.00	-0.256	1.000	1.021	0.000

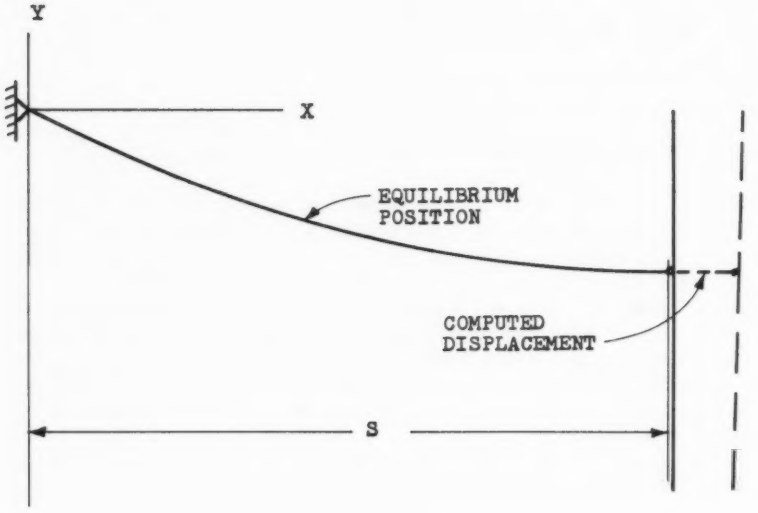

FIG. 6.—EXAMPLE 1

Q = 10.55 LBS/FT
H = 2.50 KIPS
HOR SPAN = S = 950.00 FT
HEIGHT = 1349.00 FT
PERIOD = 300.00 SEC

ALPHA = 0.0209 RADIANS/SEC
A1 = -1.2866 A2/S = -0.4860
GAMMA 0 = -1.0318 GAMMA 1 = 1.4395
R = 2.0045 BETA = 0.2279

X/S	Y/S (EQ. 4)	T/H (EQ. 7)	D (EQ. 49 INTEG)	ETA* (EQ. 61)
0.00	0.000	1.948	0.000	0.000
0.10	-0.132	1.419	0.129	0.137
0.20	-0.207	1.120	0.241	0.272
0.30	-0.236	1.004	0.343	0.399
0.40	-0.224	1.051	0.444	0.510
0.50	-0.169	1.269	0.551	0.603
0.60	-0.063	1.694	0.671	0.683
0.70	0.111	2.395	0.813	0.757
0.80	0.383	3.486	0.983	0.832
0.90	0.797	5.144	1.189	0.909
1.00	1.420	7.641	1.440	0.991

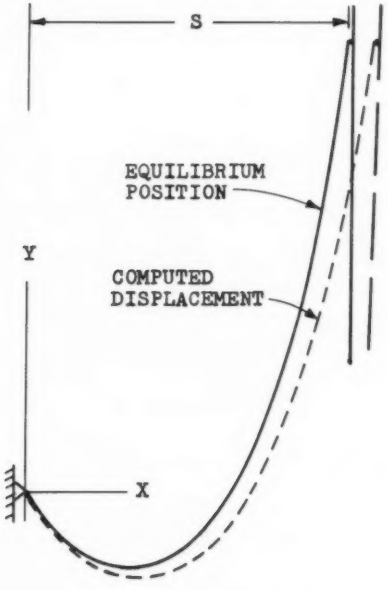

FIG. 7.—EXAMPLE 2

Q = 10.55 LBS/FT
H = 10.00 KIPS
HOR SPAN = S = 950.00 FT
HEIGHT = 0.00 FT
PERIOD = 600.00 SEC

ALPHA = 0.0105 RADIANS/SEC
A1 = -0.5011 A2/S = -1.1257
GAMMA 0 = -0.4814 GAMMA 1 = 0.4814
R = 0.5011 BETA = 0.0570

X/S	Y/S (EQ. 4)	T/H (EQ. 7)	D (EQ. 49 INTEG)	ETA* (EQ. 61)
0.00	0.000	1.128	0.000	0.000
0.10	-0.047	1.081	0.105	0.048
0.20	-0.082	1.046	0.208	0.096
0.30	-0.108	1.020	0.310	0.144
0.40	-0.123	1.005	0.410	0.192
0.50	-0.128	1.000	0.510	0.239
0.60	-0.123	1.005	0.610	0.285
0.70	-0.108	1.020	0.711	0.331
0.80	-0.082	1.046	0.813	0.376
0.90	-0.047	1.081	0.916	0.420
1.00	-0.000	1.128	1.021	0.463

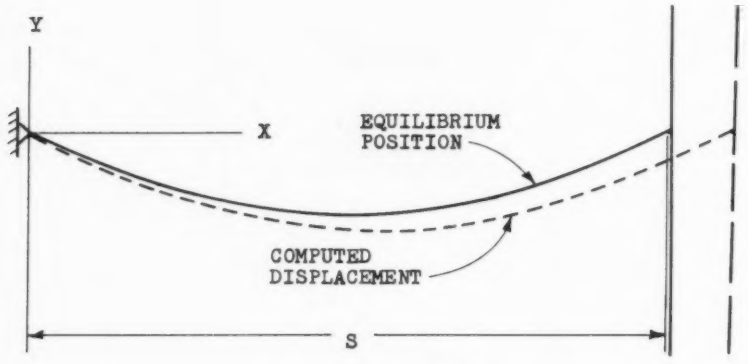

FIG. 8.—EXAMPLE 3

in which $D_{(\bar{x})}$ is clearly non-negative and bounded and independent of α. Inasmuch as, from Eq. 45,

$$\beta^2 = \frac{2\,l}{g}\ r\ \alpha^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (63)$$

in which g is the acceleration of gravity, β can be made arbitrarily small by making the support oscillation sufficiently slow. Now if r is large (small horizontal component of static cable tension) the cable sag at equilibrium will be large. If the support oscillation is sufficiently slow; more specifically, if

$$\alpha < \pi \sqrt{\frac{g}{2\,r\,l\,\cosh\left(2\,r + |a_1|\right)}} \quad \ldots\ldots\ldots \quad (64)$$

then sin β $D_{(\bar{x})}$ will be positive for all $\bar{x}$ in $0 < \bar{x} \leq 1$, and η^* will have the same sign as γ_1 throughout. According to this, if one took a cable with a large initial sag and slowly moved one support horizontally to increase the distance between supports, the sag would increase--another contradiction. Figs. 7 and 8 present two such cases.

The only restriction implied in the paper is the mention of "a cable whose tension varies only slightly along its length" in the paragraph following Eq. 45, and Figs. 6 and 8 seem to meet this restriction. The writer would like to see a clarification of the dynamic analysis and a statement of any restrictions the author may have intended to impose.

ANALYSIS OF STRUCTURES BY COMBINING REDUNDANTS[a]

Discussion by Steven J. Fenves

STEVEN J. FENVES,[11] A. M. ASCE.—There has been presented a most interesting approach to the analysis of indeterminate structures. The development of the method, the physical interpretation, and the relation of the method to other techniques are clearly presented, and these make the paper easy to follow and understand. Several textbooks approach the column analogy and elastic center methods from the standpoint of orthogonalization, but because the interpretation given is always in terms of geometry, it is not pointed out that these methods can be extended to more than three redundants. The authors have made a valuable contribution in showing the generality of these methods.

Because the development of high-speed computers, there has been considerable controversy as to whether methods of analysis reducing to the solution of a set of simultaneous equations are best suited for computer programming. The method presented by Gillis and Gerstle that eliminates the solution of simultaneous equations, appears, at first sight, to be quite advantageous for computer solution. The writer undertook to compare the proposed method to the classical inversion method of analysis in terms of the number of operations (additions, multiplications, and divisions) involved. The writer found, somewhat to his surprise, that in a large class of problems, including Fig. 2, the classical method involves fewer operations, and thus is presumably faster than the proposed method, whether programmed for a computer or done manually.

Specifically, the writer undertook to solve the following problem: given a structure consisting of n members, and having m redundants (internal or external), what is the number of basic operations (additions, multiplications, and divisions) required to solve for the values of redundant x. It was assumed that the n-values of $\alpha = L/(A\,E)$, the $n \times m$ values of s_i (stresses due to unit redundants) and the n-values of s_0 (stresses due to external loads) have been computed. In the computations, the operations involved in the "bookkeeping" of the computer program were not included.

Table 2 shows the number of operations involved in the proposed method. Because multiplication and division times are of the same order of magnitude in most computers, these two operations were grouped together in the totals.

Table 3 shows the number of operations involved in the classical method. It is to be noted that the most formal approach has been used, namely setting up the matrix of flexibility coefficients (taking cognizance of symmetry), inverting the resulting matrix, and post-multiplying the inverse with the Δ-vector.

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[a] December 1960, by Peter P. Gillis and Kurt H. Gerstle (Proc. Paper 2712).

[11] Instr. in Civ. Engrg., Univ. of Illinois, Urbana, Ill.

TABLE 2.—PROPOSED METHOD[a]

Quantity	Number of times quantity evaluated	Number of operations for each quantity	Total number of operations
$\sum_{n} \alpha S_i^2$	m	n (2 M + A)	m n (2 M + A)
$\sum_{n} \alpha S_i S_j$	$\frac{(m-1)m}{2}$	n (2 M + A)	$\frac{(m-1)m\,n}{2}$ (2 M + A)
$C_{ij} = -\frac{2.}{1.}$	$\frac{(m-1)m}{2}$	D	$\frac{(m-1)m}{2}$ D
$S_i = \sum_{j=1}^{i-1} C_{ij} S_j + S_i$	m	. . .[b]	$\frac{(m-1)m\,n}{2}$ (M + A)
$\sum_{n} \alpha S_o S_i$	m	n (2 M + A)	m n (2 M + A)
$x_i = -\frac{5.}{1.}$	m	D	m D

TOTALS: Multiplication and division $n(\frac{3}{2}m^2 + \frac{5}{2}m) + \frac{1}{2}(m^2 + m)$

Addition $n(m^2 + m)$

[a] A = addition, M = multiplication, D = division
[b] Number of operations varies = n (i-1) (M + A) for S_i

TABLE 3.—INVERSION METHOD

Quantity	Number of times quantity evaluated	Number of operations for each quantity	Total number of operations
$\delta_{ij} = \sum_{n} \alpha s_i s_j$	$\frac{(m+1)m}{2}$	n (2 M + A)	$\frac{(m+1)m\,n}{2}$ (2 M + A)
$[F_{ij}] = [\delta_{ij}]^{-1}$	1	m^3 (M + A)	m^3 (M + A)
$\Delta_i = \sum_{n} \alpha S_o s_i$	m	n (2 M + A)	m n (2 M + A)
$x_i = \sum_{m} F_{ij} \Delta_j$	m	m (M + A)	m^2 (M + A)

TOTALS: Multiplication $n(m^2 + 3m) + m(m^2 + m)$

Addition $n(\frac{1}{2}m^2 + \frac{3}{2}m) + m(m^2 + m)$

Comparing the two methods, the difference in the number of operations between the classical and the proposed method is:

$$\text{Multiplication } \frac{n}{2}\left(m - m^2\right) + \left(m - \frac{1}{2}\right)\left(m^2 + m\right) \quad \ldots\ldots \quad (34)$$

$$\text{Addition } \quad \frac{n}{2}\left(m - m^2\right) + m\left(m^2 + m\right) \quad \ldots\ldots\ldots \quad (35)$$

Since normally $m >> 1$, $m - \frac{1}{2} \simeq m$, and the difference is simply

$$\left[\frac{n}{2}\left(m - m^2\right) + m\left(m^2 + m\right)\right] \text{ (multiplications + additions)} \quad \ldots\ldots \quad (36)$$

This quantity is positive if

$$n < 2m\,\frac{m+1}{m-1} \quad \ldots\ldots\ldots\ldots\ldots \quad (37)$$

For $m = 3$, Eq. 37 shows that if $n > 12$, the proposed method involves more machine operations and is presumably slower than the classical method. For the authors' example, $m = 3$, $n = 18$, so that the classical method is more efficient. It is interesting to note, that by the foregoing reasoning, the well-known methods of column analogy and elastic centers, in which $m = 3$ and $n \leq 4$, are more efficient than the inversion method.

In conclusion, this discussion is in no way intended to detract from the value of the paper. It merely attempts to show that for the common types of structures, in which the number of members greatly exceeds the number of redundants, the authors' method does not have any computational advantages.

RECENT TRENDS IN ULTIMATE STRENGTH DESIGN[a]

Discussion by Jack R. Benjamin

JACK R. BENJAMIN,[7] M. ASCE.—The author is to be congratulated on presenting a type of progress report that has been needed for some time. Large comprehensive papers require so much condensation before publication that they are often difficult to read even for experts in the field.

The role of the theory of probability in assigning load factors is uncertain at this time (1961). Statistical concepts may be used as guides. Torroja, Baker, and others have used this approach. In reality, this is little more than "lip service" to statistics. The load factors recommended remain essentially judgment items.

Alfred M. Freudenthal, F. ASCE and Oliver G. Julien have advocated a rational application of statistics to the design problem. However, Freudenthal has largely abandoned this position in his most recent writings. The problem is that there is absolutely no way that a small table of load factors can be constructed in conformity with statistical concepts. All attempts in this direction are doomed to failure. The magnitude of the problem is comparable to reducing all reinforced concrete design to a single-page publication to be used for all possible conditions.

Statistical procedures are of great value to the structural engineer. First, descriptive statistics can be used to design tests, study test results, and review the analytical design procedures produced. For example, elementary statistical procedures disclose that the formula for ultimate strength of reinforced concrete beams contains a basic error. The source of the error can be determined by regression analysis; such a study is currently in progress at Sanford University, California.

The second contribution statistics can make is in the basic reasoning leading to judgment decisions. The load factor is a typical example. Unfortunately, a surface knowledge of statistics is almost worse than no knowledge at all. One could very well make serious errors in judgment if undue weight is given to the limited and elementary treatments in vogue in the literature. The primary source of error appears to be in the statistical concepts of loads, both normal and predicted extreme loads. Simple load surveys are totally inadequate unless the underlying laws of behavior can be determined. It is evident that live and wind loads are stochastic processes requiring more than elementary statistical treatment.

Finally, statistical procedures can be developed to become a completely new and rational system for analysis and design. In such a system, all the current

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[a] January 1961, by Phil M. Ferguson (Proc. Paper 2715).

[7] Prof. of Structural Engrg., Stanford Univ., Stanford, Calif.

procedures would be abandoned for they generally are not statistically rational. Such a development is far in the future, if ever.

It cannot be emphasized too strongly that simple probability concepts are not adequate in themselves to replace basic judgment items in structural design. Every concession to probability concepts carries with it a responsibility that the decision made is statistically valid.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Department of Conditions of Practice are identified by the symbols (PP). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper number are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper 2703 is identified as 2703(ST1) which indicates that the paper is contained in the first issue of the Journal of the Structural Division during 1961.

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APRIL: 2428(ST4), 2429(HY4), 2430(PO2), 2431(SM2), 2432(PO2), 2433(ST4), 2434(EM2), 2435(PO2), 2436 (ST4), 2437(ST4), 2438(HY4), 2439(EM2), 2440(EM2), 2441(ST4), 2442(SM2), 2443(HY4), 2444(ST4), 2445 (EM2), 2446(ST4), 2447(EM2), 2448(SM2), 2449(HY4), 2450(ST4), 2451(HY4), 2452(HY4), 2453(EM2), 2454 (EM2), 2455(EM2)[c], 2456(HY4)[c], 2457(PO2)[c], 2458(ST4)[c], 2459(SM2)[c].

MAY: 2460(AT1), 2461(ST5), 2462(AT1), 2463(AT1), 2464(CP1), 2465(CP1), 2466(AT1), 2467(AT1), 2468(SA3), 2469(HY5), 2470(ST5), 2471(SA3), 2472(SA3), 2473(ST5), 2474(SA3), 2475(ST5), 2476(SA3), 2477(ST5), 2478 (HY5), 2479(SA3), 2480(ST5), 2481(SA3), 2482(CO2), 2483(CO2), 2484(HY5), 2485(HY5), 2486(AT1)[c], 2487 (CP1)[c], 2488(CO2)[c], 2489(HY5)[c], 2490(SA3)[c], 2491(ST5)[c], 2492(CP1), 2493(CO2).

JUNE: 2494(IR2), 2495(IR2), 2496(ST6), 2497(EM3), 2498(EM3), 2499(EM3), 2500(EM3), 2501(SM3), 2502 (EM3), 2503(PO3), 2504(WW2), 2505(EM3), 2506(HY6), 2507(WW2), 2508(PO3), 2509(ST6), 2510(EM3), 2511 (EM3), 2512(ST6), 2513(HW2), 2514(HY6), 2515(PO3), 2516(EM3), 2517(WW2), 2518(WW2), 2519(EM3), 2520 (PO3), 2521(HY6), 2522(SM3), 2523(ST6), 2524(HY6), 2525(HY6), 2526(HY6), 2527(IR2), 2528(ST6), 2529 (HW2), 2530(IR2), 2531(HY6), 2532(EM3)[c], 2533(HW2)[c], 2534(WW2), 2535(HY6)[c], 2536(IR2)[c], 2537(PO3)[c], 2538(SM3)[c], 2539(ST6)[c], 2540(WW2)[c].

JULY: 2541(ST7), 2542(ST7), 2543(SA4), 2544(ST7), 2545(ST7), 2546(HY7), 2547(ST7), 2548(SU2), 2549(SA4), 2550(SU2), 2551(HY7), 2552(ST7), 2553(SU2), 2554(SA4), 2555(ST7), 2556(SA4), 2557(SA4), 2558(SA4), 2559(ST7), 2560(SU2)[c], 2561(SA4)[c], 2562(HY7)[c], 2563(ST7)[c].

AUGUST: 2564(SM4), 2565(EM4), 2566(ST8), 2567(EM4), 2568(PO4), 2569(PO4), 2570(HY8), 2571(EM4), 2572(EM4), 2573(EM4), 2574(SM4), 2575(EM4), 2576(EM4), 2577(HY8), 2578(EM4), 2579(PO4), 2580 (EM4), 2581(ST8), 2582(ST8), 2583(EM4)[c], 2584(PO4)[c], 2585(ST8)[c], 2586(SM4)[c], 2587(HY8)[c].

SEPTEMBER: 2588(IR3), 2589(IR3), 2590(WW3), 2591(IR3), 2592(HW3), 2593(IR3), 2594(IR3), 2595(IR3), 2596 (HW3), 2597(WW3), 2598(IR3), 2599(WW3), 2600(WW3), 2601(WW3), 2602(WW3), 2603(WW3), 2604(HW3), 2605(SA5), 2606(WW3), 2607(SA5), 2608(ST9), 2609(SA5)[c], 2610(IR3), 2611(WW3)[c], 2612(ST9)[c], 2613(IR3)[c], 2614(HW3)[c].

OCTOBER: 2615(EM5), 2616(EM5), 2617(ST10), 2618(SM5), 2619(EM5), 2620(EM5), 2621(ST10), 2622(EM5), 2623(SM5), 2624(EM5), 2625(SM5), 2626(SM5), 2627(EM5), 2628(EM5), 2629(ST10), 2630(ST10), 2631(PO5)[c], 2632(EM5)[c], 2633(ST10), 2634(ST10), 2635(ST10)[c], 2636(SM5)[c].

NOVEMBER: 2637(ST11), 2638(ST11), 2639(CO3), 2640(ST11), 2641(SA6), 2642(WW4), 2643(ST11), 2644(HY9), 2645(ST11), 2646(HY9), 2647(WW4), 2648(WW4), 2649(WW4), 2650(ST11), 2651(CO3), 2652(HY9), 2653(HY9), 2654(ST11), 2655(HY9), 2656(HY9), 2657(SA6), 2658(WW4), 2659(WW4)[c], 2660(SA6), 2661(CO3), 2662(CO3), 2663(SA6), 2664(CO3)[c], 2665(HY9)[c], 2666(SA6)[c], 2667(ST11)[c].

DECEMBER: 2668(ST12), 2669(IR4), 2670(SM6), 2671(IR4), 2672(IR4), 2673(IR4), 2674(ST12), 2675 (EM6), 2676(IR4), 2677(HW4), 2678(ST12), 2679(EM6), 2680(ST12), 2681(SM6), 2682(IR4), 2683(SM6), 2684(SM6), 2685(IR4), 2686(EM6), 2687(EM6), 2688(EM6), 2689(EM6), 2690(EM6), 2691(EM6)[c], 2692 (ST12), 2693(ST12), 2694(HW4)[c], 2695(IR4)[c], 2696(SM6)[c], 2697(ST12)[c].

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FEBRUARY: 2726(WW1), 2727(EM1), 2728(EM1), 2729(WW1), 2730(WW1), 2731(EM1), 2732(SM1), 2733(WW1), 2734(SM1), 2735(EM1), 2736(EM1), 2737(PL1), 2738(PL1), 2739(PL1), 2740(PL1), 2741(EM1), 2742(ST2), 2743(EM1), 2744(WW1), 2745(WW1), 2746(SM1), 2747(WW1), 2748(EM1), 2749(WW1), 2750(WW1)[c], 2751 (EM1)[c], 2752(SM1)[c], 2753(PL1)[c], 2754(ST2)[c], 2755(PL1).

MARCH: 2756(HY2), 2757(IR1), 2758(AT1), 2759(CO1), 2760(HY2), 2761(IR1), 2762(IR1), 2763(HY2), 2764 (ST3), 2765(HY2), 2766(HW1), 2767(SA2), 2768(CO1), 2769(IR1), 2770(HY2), 2771(SA2), 2772(HY2), 2773 (CO1), 2774(AT1), 2775(IR1), 2776(HY2), 2777(HY2), 2778(SA2), 2779(ST3), 2780(HY2), 2781(HY2)[c], 2782 (HW1)[c], 2783(SA2)[c], 2784(CO1), 2785(CO1)[c], 2786(IR1)[c], 2787(ST3)[c], 2788(AT1)[c], 2789(HW1).

APRIL: 2790(EM2), 2791(SM2), 2792(SM2), 2793(SM2), 2794(SM2), 2795(SM2), 2796(SM2), 2797(SM2), 2798 (EM2), 2799(EM2), 2800(EM2), 2801(EM2), 2802(ST4), 2803(EM2)[c], 2804(SM2)[c], 2805(ST4)[c].

c. Discussion of several papers, grouped by divisions.

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